Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 235 (2007) 166–198

Journal of Differential Equations

www.elsevier.com/locate/jde

Regularity of solutions for the Navier–Stokes system of incompressible flows on a polygon ☆

Jae Ryong Kweon

Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Kyoungbuk, South Korea

Received 28 March 2006; revised 13 December 2006

Available online 28 December 2006

Abstract

We are concerned with singularities and regularities of solutions for the Navier–Stokes system of incompressible flows on a polygonal domain with a concave vertex. We subtract the corner singularities by the Stokes operator from the solution velocity and pressure functions of the system. It is shown that the stress intensity factors are functions of time variable, belong to a fractional Sobolev space on the time interval and can be expressed in terms of given data. An increased regularity for the remainder is obtained.

MSC: 35M10; 35B25

Keywords: Navier–Stokes; Incompressibility; Corner singularity

1. Introduction and main results

A complete mathematical analysis for singular behaviors and regularities of solutions for the Navier–Stokes system in domains with singular boundaries has not been appeared yet. Practically the system is often considered in domains having corners or edges, e.g., consider fluid mechanics in the driven cavity [29] or partitions of domains in numerical simulations.

There are several known results for stationary Stokes problems in domains with corners: in [14] a regularity result is given for the system with nonzero divergence system on convex polygons; in [4] the H^s-regularity is studied for the nonzero divergence system on domains with

☆ This work was supported by the Korea Research Foundation Grant funded by Korea Government (KRF-2005-015-C00050) and partially supported by grant No. R01-2006-000-10002-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

E-mail address: kweon@postech.ac.kr.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.12.008

corners, several examples of domains are illustrated, giving some conditions on the exponent s, and the result [14] is generalized to a result on a convex polyhedron. For a detailed description of the exponents of corner singularities of the Stokes operator we refer to [19]. In particular, in [5–7] the Stokes system or Stokes system with parameter are studied in three-dimensional domains with conical boundary points. Finally we refer to [10,15,19,28] for elliptic equations, [8,11–13,16–18,25,26,30] for the heat equation and [20–22,27] for compressible flows.

In this paper the Navier–Stokes system is considered on a polygonal domain having only one concave corner. Our main issue is to show that if we write the solution in a decomposition of singular and regular parts, then the coefficients of the singularities, called the stress intensity functions, can be expressed in terms of given data, belong to a fractional Sobolev space on the time interval, and the remainder satisfies an increased regularity. The Navier–Stokes system to be considered in this paper is

$$\begin{aligned} \partial_t \mathbf{u} - \mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } Q, \\ \operatorname{div}\mathbf{u} = 0 \quad & \text{in } Q, \\ \mathbf{u} = 0 \quad & \text{on } \Sigma, \\ \mathbf{u}(\cdot, 0) = 0 \quad & \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $Q = \Omega \times [0, \mathrm{T}]$ with a number $\mathrm{T} > 0$; Ω is a bounded plane polygonal domain whose boundary Γ has only one concave corner, $\Sigma = \Gamma \times [0, \mathrm{T}]$ is the lateral boundary of Q; $\mathbf{u}$ is the fluid velocity vector and p is the fluid pressure; $\mathbf{f}$ is a given vector function; μ is the viscous number with $\mu > 0$; ∂_t denotes the partial derivative with respect to the time variable, Δ the Laplace operator, ∇ the gradient and div the divergence for the space variable. For simplicity it is assumed that $\mu = 1$.

Our reason considering a polygonal domain having only one concave corner is for giving a precise and simple description of corner singularities for the solution of the system. In addition, problems concerning corner singularities and their stress intensity factors are important in the computational fluid mechanics [2,24,29,32]. Not only stresses and pressure singularities but also many interesting physical phenomena occur around corners, e.g., cavity problem, eddy, recirculation, flow separation and discontinuity, etc. Hence it is worthwhile to give a rigorous mathematical analysis for the singularities of solutions.

In [11–13] Grisvard studied the Laplace problems (with parameter) on bounded domains with corners and showed a corner singularity expansion for the solution of the heat equation [12]. For the Navier–Stokes system one not only has to handle several technical difficulties in deriving such a singular expansion but also can observe important and qualitative properties (e.g., singularities and regularities) of solutions of the nonlinear system. Main differences are: first, the strength of the leading corner singularities of the Stokes operator is stronger than the one of the Laplace operator and for twice differentiability of solution, one or two leading corner singularities must be subtracted (see (1.5)–(1.6)); second, suitable fractional order Sobolev spaces must be chosen in handling the singularities; third, how one can define the stress intensity factors for the nonlinear system; fourth, how one can express the stress intensity factor in terms of known data, and show its well definedness, etc. These issues will be resolved in this paper.

In this paper we often write $[v(t)](\mathbf{x}) := v(\mathbf{x}, t)$ as a mapping $t \in [0, \mathrm{T}] \mapsto v(t) \in \mathrm{X}$ a space and $'$ the differential of time variable. The derivative ∂_t is a unbounded operator on $\mathrm{L}^2(0, \mathrm{T}; \mathrm{X})$ with domain $\mathrm{H}^1(0, \mathrm{T}; \mathrm{X})$. The spectrum of ∂_t is the set of all complex numbers with $\mathcal{R}e\,\lambda \geqslant 0$

and the resolvent of ∂_t satisfies $\|(\lambda I - \partial_t)^{-1}\| \leqslant |\mathcal{R}e\,\lambda|^{-1}$ for $\mathcal{R}e\,\lambda < 0$. We will consider the operator $\mathcal{T}$ defined by $\mathcal{T}[\mathbf{v}, q] = [\partial_t \mathbf{v}, 0]$.

In this paper we consider the following spaces [1,3,9,23,31]. For subsets $\mathcal{O} \subset \Omega$, $\mathrm{H}^s(\mathcal{O})$ and $\|u\|_{s,\mathcal{O}}$ denote the Sobolev space of order $s \geqslant 0$ and the corresponding norm. We write $\mathrm{H}^s = \mathrm{H}^s(\Omega)$ and $\|u\|_s = \|u\|_{s,\Omega}$ if $\mathcal{O} = \Omega$. For $s > 0$, H_0^s denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ in H^s and H^{-s} the dual space of H_0^s normed by

$$\|f\|_{-s} = \sup_{0 \neq v \in \mathrm{H}_0^s} \frac{\langle f, v\rangle}{\|v\|_s}$$

where $\langle\,,\rangle$ denotes the duality pairing. In particular, we write $\mathrm{L}^2 = \mathrm{H}^0$ and $\mathrm{L}_0^2 = \{q \in \mathrm{L}^2\colon \int_\Omega q\,\mathrm{d}\mathbf{x} = 0\}$. We write $\mathbf{H}^s = \mathrm{H}^s \times \mathrm{H}^s$, $\mathbf{L}^2 = \mathrm{L}^2 \times \mathrm{L}^2$ and so on. We set $\mathcal{K} = \{\mathbf{v} \in \mathcal{C}_0^\infty(\Omega);\ \operatorname{div} \mathbf{v} = 0\}$ and, for real s,

$$\begin{aligned} \mathrm{V}^s &= \text{the closure of } \mathcal{K} \text{ in } \mathbf{H}^s \cap \mathbf{H}_0^1, \\ \mathrm{V}^{-s} &= \text{the dual space of } \mathrm{V}^s, \\ \mathrm{H} &= \text{the closure of } \mathcal{K} \text{ in } \mathbf{L}^2. \end{aligned} \tag{1.2}$$

Write $\mathrm{V} = \mathrm{V}^1$ and $\mathrm{H}^{-1} =$ the dual space of H. For a space X with norm $\|\cdot\|$, $\mathrm{L}^2(0, \mathrm{T}; \mathrm{X})$ is the space of measurable functions $v : [0, \mathrm{T}] \mapsto \mathrm{X}$, satisfying $\|v\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{X})} := (\int_0^{\mathrm{T}} \|v(t)\|^2\,\mathrm{d}t)^{1/2} < \infty$ and $\mathrm{L}^\infty(0, \mathrm{T}; \mathrm{X})$ is the space of measurable functions $v : [0, \mathrm{T}] \mapsto \mathrm{X}$, satisfying $\|v\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{X})} := \operatorname{ess\,sup}_{0 \leqslant t \leqslant \mathrm{T}} \|v(t)\| < \infty\}$. Also $\mathrm{L}^\infty(Q) = \{v\colon \|v\|_{\infty,Q} < \infty\}$ where $\|v\|_{\infty,Q} := \operatorname{ess\,sup}_{(\mathbf{x},t)\in Q} |v(\mathbf{x}, t)|$. In particular, $\mathrm{H}^1(0, \mathrm{T}; \mathrm{X}) = \{v\colon v, v' \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{X})\}$ and for $0 < s < 1$, $\mathrm{H}^s(0, \mathrm{T}; \mathrm{X})$ denotes the interpolation space between integers 0, 1. We often use the embedding result: $\mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H}^s(\Omega)) \subset \mathrm{L}^\infty(Q)$ for $s > 1$.

Here we cite from Temam [31] a well-known regularity result for (1.1).

Theorem 1.1. *Let Ω be a bounded Lipschitz plane domain.* (i) *If* $\mathbf{f} \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{V}^{-1})$*, then there is a unique solution* $[\mathbf{u}, p]$ *of* (1.1) *in the space* $\mathrm{L}^2(0, \mathrm{T}; \mathrm{V}) \times \mathrm{L}^2(0, \mathrm{T}; \mathrm{L}_0^2)$. (ii) *If* $\mathbf{f} \in \mathrm{H}^1(0, \mathrm{T}; \mathrm{V}^{-1})$*, then* $\mathbf{u}' \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{V}) \cap \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H})$ *and* $p \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{L}_0^2)$. (iii) *Let Ω be a bounded plane domain of class $\mathcal{C}^2$. If* $\mathbf{f} \in \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H}) \cap \mathrm{H}^1(0, \mathrm{T}; \mathrm{V}^{-1})$*, then* $\mathbf{u} \in \mathrm{L}^\infty(0, \mathrm{T}; \mathbf{H}^2)$ *and* $p \in \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H}^1)$.

Proof. The proof follows by Theorems 3.1–3.2 and 3.5–3.6 in Temam [31, Chapter III, Section 3]. □

In this paper an issue to be resolved is to extend the regularity result (iii) of Theorem 1.1 to the polygonal domain case.

Let P be the concave vertex of the domain Ω, placed at the origin. The geometry of Ω near the corner P is as follows. Let ω_1 and ω_2 be numbers satisfying $-\pi < \omega_1 < 0 < \omega_2 < \pi$. We assume that in a neighborhood of P, Ω coincides with the sector

$$\mathcal{S} = \big\{(r\cos\theta, r\sin\theta)\colon\ \omega_1 < \theta < \omega_2,\ 0 < r < \infty\big\}$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. The angle of the polygon at the origin is $\omega = \omega_2 - \omega_1 > \pi$. Let $\alpha = \pi/\omega$. We assume that $\omega \neq 2\pi$. We now describe the formula of the singular functions for the Stokes operator in the sector $\mathcal{S}$ and define numbers for the regularities of the singular functions. To describe these formula, consider the transcendental equation in [19]:

$$\sin^2(\lambda\omega) - \lambda^2 \sin^2\omega = 0.$$

This equation has an infinite number of complex solutions. Ordering these solutions with non-decreasing real part, we get a nondecreasing sequence of numbers λ_i [19]:

$$1/2 < \lambda_1 < \alpha < \mathcal{R}\mathrm{e}\,\lambda_2 < \mathcal{R}\mathrm{e}\,\lambda_3 < 2\alpha < \cdots.$$

The numbers s_i are given by

$$s_i = \mathcal{R}\mathrm{e}\,\lambda_i + 1. \tag{1.3}$$

The singular functions Φ_i and ϕ_i corresponding to the velocity and the pressure have the forms

$$\Phi_i = \chi r^{\lambda_i} \mathcal{T}_i(\theta), \qquad \phi_i = \chi r^{\lambda_i - 1} \xi_i(\theta) \tag{1.4}$$

where $[\mathcal{T}_i, \xi_i]$ is the eigenvector corresponding to λ_i, $\mathcal{T}_i$ is a trigonometric vector function with $\mathcal{T}_i(\omega_1) = \mathcal{T}_i(\omega_2) = 0$ and ξ_i is a trigonometric scalar function, and χ is a smooth cutoff function near the origin.

Some more information is available concerning the numbers λ_i [19]. If the vertex P is convex, so $\omega < \pi$, then $\lambda_1 = 1$ is the unique and simple eigenvalue in the strip $0 < \mathcal{R}\mathrm{e}\,\lambda < \alpha$ and its corresponding velocity eigenfunction $\mathcal{T}_i$ is zero and the pressure eigenfunction ξ_i is constant. If the vertex P is concave, so $\omega > \pi$, then the first 3 roots, λ_i for $i = 1, 2, 3$, are real and satisfy the inequalities

$$1/2 < \lambda_1 < \alpha < \lambda_2 = 1 < \lambda_3 < 2\alpha, \quad \omega \in (\pi, \omega_*), \tag{1.5}$$

$$1/2 < \lambda_1 < \alpha < \lambda_2 < \lambda_3 = 1 < 2\alpha, \quad \omega \in (\omega_*, 2\pi), \tag{1.6}$$

where ω_* is the unique solution of the equation $\tan\omega = \omega$ in the interval $(0, 2\pi]$, in fact $\omega_* \sim 1.4303\pi$.

From (1.5)–(1.6) we have $\lambda_2 = 1$ for $\omega \in (\pi, \omega_*)$ and $\lambda_3 = 1$ for $\omega \in (\omega_*, 2\pi)$. So the singular function $[\Phi_i, \phi_i]$ corresponding to this eigenvalue 1 does not have to be split from the solution. Hence the order s of the Sobolev space H^s is chosen in the interval $s_1 < s \leqslant 2 < s_3$ for $\omega \in (\pi, \omega_*)$ and $s_2 < s \leqslant 2 < s_4$ for $\omega \in (\omega_*, 2\pi)$. Throughout this paper we define $N = 1$ for $\omega \in (\pi, \omega_*)$, $N = 2$ for $\omega \in (\omega_*, 2\pi)$ and set

$$\mathcal{E}(r, t) = \frac{1}{\sqrt{4\pi}} t^{-3/2} r e^{-r^2/4t}, \qquad (u \star v)(t) = \int_0^\infty u(t - s) v(s)\,\mathrm{d}s. \tag{1.7}$$

Here we give the main result of this paper, which is shown in Section 4.

Theorem 1.2. *Let Ω be a bounded polygon having only one concave vertex placed at the origin. Let $N = 1$ for $\omega \in (\pi, \omega_*)$ and $N = 2$ for $\omega \in (\omega_*, 2\pi)$. Let $s_N < s \leqslant 2$ be given. Assume that* $\mathbf{f} \in \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{V}^{s-2}) \cap \mathrm{H}^1(0, \mathrm{T}; \mathrm{V}^{-1})$. *If we write the solution* $[\mathbf{u}, p]$ *of* (1.1) *in the form*

$$[\mathbf{u}, p] = \sum_{j=1}^{N} \big(\mathcal{E}(r, \cdot) \star c_j\big)[\Phi_j, \phi_j] + [\mathbf{u}_R, p_R], \tag{1.8}$$

then the remainder $[\mathbf{u}_R, p_R] \in \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{V}^s) \times \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H}^{s-1})$*, and the coefficient function* c_j *can be written by*

$$c_j(t) = \frac{1}{2\pi i} \int_\gamma \big\langle \Lambda_j(\lambda); (\lambda I - \mathcal{T})^{-1} \eta_j(t) \big\rangle \mathrm{d}\lambda, \tag{1.9}$$

$\Lambda_j(\lambda)$ *is defined in* (2.12) *and a continuous linear functional on* $\mathrm{V}^{s-2} \times \mathrm{H}^{s-1}$ *for* $s > s_j$*,* η_j *is a vector function of the form given in* (3.35) *and* (3.38)*, where* $i = \sqrt{-1}$ *and* γ *is a vertical axis satisfying* $\mathcal{R}\mathrm{e}\,\lambda < 0$, $\lambda \in \gamma$*. Furthermore the function* $c_j \in \mathrm{H}^{(s-s_j)/2}(0, \mathrm{T})$ *for* $s > s_j$*,* $c_j = 0$ *outside* $[0, \mathrm{T}]$*, and the remainder* $[\mathbf{u}_R, p_R]$ *satisfies the a priori estimate*

$$\begin{aligned} &\|\mathbf{u}_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{H}^s)} + \|p_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}^{s-1})} + \|\mathbf{u}'_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \sum_{j=1}^{N} \|c_j\|_{\mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})} \\ &\quad \leqslant C\big(\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\big), \end{aligned} \tag{1.10}$$

where $C = C(\mathrm{T})$*. If the domain* Ω *is convex, then* $[\mathbf{u}, p]$ *satisfies* (1.10).

From Theorem 1.2 we see that the solution of problem (1.1) can have a corner singularity expansion near concave corners, and if $s = 2$ is chosen, the remainder has the same regularity as shown in the smooth domain (see the case (iii) of Theorem 1.1), and that the coefficient function, called the stress intensity function, is well defined, can be expressed by a contour integral of well-defined functions (for a detail, see Step 3 in the proof of Theorem 3.1). See [11] for the contour integral of (1.9). The unique existence of solution of (1.1) can be shown by the same method as given in the proof of Theorem 1.1. The formula $\Lambda_j(\lambda)$ is a functional for the stress intensity factor for the Stokes operator with parameter λ (see (2.12)) and the function c_j in (1.9) is the stress intensity function defined on the time interval, which is derived in (3.20) and Step 3 given in the proof of Theorem 3.1.

In order to show Theorem 1.2 we consider a linearized version of system (1.1). Let $\mathbf{w}$ be a given vector function, with $\mathbf{w}|_\Sigma = 0$ and $\mathbf{w}(\cdot, 0) = 0$. Then a linearized system of (1.1) reads

$$\begin{aligned} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} &\quad \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 &\quad \text{in } Q, \\ \mathbf{u} = 0 &\quad \text{on } \Sigma, \\ \mathbf{u}(\cdot, 0) = 0 &\quad \text{in } \Omega. \end{aligned} \tag{1.11}$$

We define the Stokes operator L and the $\mathbf{w}$-directional derivative operator $B_\mathbf{w}$ as follows:

$$\mathrm{L}[\mathbf{v}, q] := [-\Delta \mathbf{v} + \nabla q, \operatorname{div} \mathbf{v}], \qquad B_\mathbf{w}[\mathbf{v}, q] := [\mathbf{w} \cdot \nabla \mathbf{v}, 0]. \tag{1.12}$$

Using the operators $\mathcal{T}$, L and $B_\mathbf{w}$, system (1.11) can be written as follows:

$$\begin{aligned}(\mathcal{T}+\mathrm{L}+B_{\mathbf{w}})[\mathbf{u},p] &= [\mathbf{f},0] \quad \text{in } Q,\\ \mathbf{u} &= 0 \quad \text{on } \Sigma,\\ \mathbf{u}(\cdot,0) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{1.13}$$

This formulation will be used later, for example, in expressing the stress intensity function in terms of known data (see Step 3 in the proof of Theorem 3.1).

Next we give a result of the corner singularity expansion for the linear problem (1.11), which is proved in Section 3.

Theorem 1.3. *Let Ω have only one concave corner, placed at the origin. Suppose that $\mathbf{w} \in \mathrm{L}^\infty(Q)$ and $\mathbf{w}' \in \mathrm{L}^2(0,\mathrm{T};\mathrm{V}) \cap \mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})$.* (i) *If $\mathbf{f} \in \mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2}) \cap \mathrm{H}^{s-1}(0,\mathrm{T};\mathrm{V}^{-1})$ for $1 \leqslant s < s_1$, then there is a unique solution $[\mathbf{u},p]$ of* (1.11)*, satisfying*

$$\begin{aligned}&\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^s)} + \|p\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{s-1})} + \|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}\\ &\quad \leqslant C\big(\|\mathbf{f}\|_{\mathrm{H}^{s-1}(0,\mathrm{T};\mathrm{V}^{-1})} + \|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}\big),\end{aligned} \tag{1.14}$$

where $C = C(\mathrm{T}, \|\mathbf{w}\|_{\infty,Q}, \|\mathbf{w}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}, \|\mathbf{w}'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})})$. (ii) *Let $s_N < s \leqslant 2$ be given. Assume that $\mathbf{f} \in \mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2}) \cap \mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})$. If we write the solution $[\mathbf{u},p]$ in the form*

$$\big[\mathbf{u}(t), p(t)\big] = \sum_{j=1}^{N} (\mathcal{E} \star c_j)(t)[\Phi_j, \phi_j] + \big[\mathbf{u}_R(t), p_R(t)\big], \tag{1.15}$$

then the function $c_j \in \mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})$ for $s > s_j$ and is given by

$$c_j(t) = \frac{1}{2\pi i} \int_\gamma \big\langle \Lambda_j(\lambda); (\lambda I - \mathcal{T})^{-1} \eta_j(t) \big\rangle \, \mathrm{d}\lambda, \tag{1.16}$$

η_j is the vector function given in (3.35) *and* (3.38)*, where γ is a vertical axis satisfying $\mathcal{R}\mathrm{e}\,\lambda < 0$, $\lambda \in \gamma$, and the remainder $[\mathbf{u}_R, p_R]$ satisfies*

$$\begin{aligned}&\|\mathbf{u}_R\|_{\mathrm{L}^2(0,\mathrm{T};\mathbf{H}^s)} + \|p_R\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{s-1})} + \|\mathbf{u}'_R\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})} + \sum_{j=1}^{N} \|c_j\|_{\mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})}\\ &\quad \leqslant C\big(\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})} + \|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\big),\end{aligned} \tag{1.17}$$

where $C = C(\mathrm{T}, \|\mathbf{w}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}, \|\mathbf{w}'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})})$. Furthermore, if we assume that $\mathbf{f} \in \mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})$, then

$$\begin{aligned}&\|\mathbf{u}_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{H}^s)} + \|\mathbf{u}'_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \|p_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}^{s-1})} + \sum_{j=1}^{N} \|c_j\|_{\mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})}\\ &\quad \leqslant C\big(\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\big),\end{aligned} \tag{1.18}$$

where $C = C(\mathrm{T}, \|\mathbf{w}\|_{\infty,Q}, \|\mathbf{w}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}, \|\mathbf{w}'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})})$. *If the domain* Ω *is convex, then* $[\mathbf{u}, p] = [\mathbf{u}_R, p_R]$ *satisfies the inequalities* (1.17) *and* (1.18).

In Theorem 1.3 the estimate (1.14), which is shown in Lemma 3.1, is a (maximal) regularity that the solution of the linear problem (1.11) can have on the concave polygon Ω, without subtracting corner singularities. The assumption on $\mathbf{w}$ is $\mathbf{w}' \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{V}) \cap \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H})$, which corresponds to the inequality (3.3) for $\mathbf{u}$ and the inequality (3.22) for the regular part $\mathbf{u}_R$. Also the condition $\mathbf{w} \in \mathrm{L}^\infty(Q)$ corresponds to the regularity $\mathbf{u}_R \in \mathrm{L}^\infty(0, \mathrm{T}; \mathbf{H}^s)$ for $s > 1$, in fact, $s \in (s_N, 2]$. For handling the nonlinearity we will split the fixed vector $\mathbf{w}$ into singular and regular parts (see (4.4)). The estimate (1.18) is used in solving the nonlinear problem (1.1).

The Laplace transform, denoted by $\mathcal{L}$, is defined by

$$\hat{u}(z) := \mathcal{L}\{u(t)\}(z) = \int_0^\infty e^{-zt} u(t)\,\mathrm{d}t, \quad \xi := \mathcal{R}\mathrm{e}\, z > 0,$$

and its inverse Laplace transform $\mathcal{L}^{-1}$ is defined by

$$u(t) := \mathcal{L}^{-1}\{\hat{u}(z)\}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{zt} \hat{u}(z)\,\mathrm{d}z \quad (c \text{ is fixed}).$$

Throughout this paper C denotes a generic positive constant and may take different values in different places.

This paper is organized as follows. In Section 2 we define the stress intensity factors of corner singularities, estimate them (see Lemmas 2.2–2.4) and derive a basic result of corner singularity for the Stokes problem with parameter (see Theorem 2.2). In Section 3 we derive a corner singularity expansion for a linearized Navier–Stokes system and estimate the remainder (see Theorem 3.1). In Section 4 we show Theorem 1.2 for the nonlinear problem (1.1) by constructing a fixed point mapping.

2. The Stokes problem with parameter

We show a basic result of corner singularity for the stationary Stokes problem (2.1) with complex parameter on the polygon Ω (see Theorem 2.2). In the result we construct the stress intensity factors depending on the parameter, subtract corner singularities depending on the parameter from the solution and show an increased regularity for the remainder. This result will be used in deriving the corner singularity result for the evolution Stokes problem (1.11).

First we state a basic result of the corner singularity to the stationary Stokes problem, which can be derived from [10,19] and the interpolation theory [23].

Theorem 2.1. *Let* Ω *have only one concave corner, placed at the origin. Suppose* $\mathbf{f} \in \mathrm{V}^{-1}$. *Then there is a unique solution* $[\mathbf{u}, p] \in \mathrm{V} \times \mathrm{L}_0^2$ *for the Stokes problem*: $\mathrm{L}[\mathbf{u}, p] = [\mathbf{f}, 0]$ *in* Ω *and* $\mathbf{u} = 0$ *on* Γ. *If* $1 \leqslant s < s_1$ *and* $\mathbf{f} \in \mathrm{V}^{s-2}$, *then* $\|\mathbf{u}\|_s + \|p\|_{s-1} \leqslant C\|\mathbf{f}\|_{\mathrm{V}^{s-2}}$ *for a constant* C.

On the other hand, let $1 \leqslant i \leqslant N$ *be given where* $N = 1$ *or* 2, *depending on the angle* $\omega > \pi$ (*see* (1.5)–(1.6)). *There is a bounded linear functional* Λ_i *on* $\mathrm{V}^{s-2} \times \mathrm{H}^{s-1}$, $s > s_i$, *and the* *singular function* $[\Phi_i, \phi_i]$ *given in* (1.4) *belongs to* $\mathrm{V}^s \times \mathrm{H}^{s-1}$ *for* $s < s_i$ *but not in* $\mathrm{V}^{s_i} \times \mathrm{H}^{s_i-1}$. *If* $\mathbf{f} \in \mathrm{V}^{s-2}$ *for* $s > s_N$, *then the solution can be split as follows*:

$$[\mathbf{u}, p] = \sum_{i=1}^{N} \Lambda_i(\mathbf{f}, 0)[\Phi_i, \phi_i] + [\mathbf{u}_R, p_R],$$

with $[\mathbf{u}_R, p_R] \in \mathbf{H}^s \times \mathrm{H}^{s-1}$. *Also, if* $\mathbf{f} \in \mathrm{V}^{s-2}$ *for* $s > s_N$, *the remainder* $[\mathbf{u}_R, p_R]$ *and the stress intensity coefficients* $\Lambda_i(\mathbf{f}, 0)$ *satisfy the a priori estimate*

$$\|\mathbf{u}_R\|_s + \|p_R\|_{s-1} + \sum_{i=1}^{N} \big|\Lambda_i(\mathbf{f}, 0)\big| \leqslant C\|\mathbf{f}\|_{\mathrm{V}^{s-2}}$$

where C *is a generic constant.*

Now we are going to establish a regularity result like Theorem 2.1 for the Stokes problem with parameter (see Theorem 2.2). For certain related references one may refer to [6,7], in which the Stokes system or the Stokes system with parameter were studied in three-dimensional domains with conical boundary points but the approaches are different.

The Stokes problem with parameter is defined as follows: given complex number ζ with $\mathcal{R}\mathrm{e}\,\zeta > 0$, find $[\mathbf{u}, p] \in \mathrm{V} \times \mathrm{L}_0^2$ such that

$$\begin{aligned} (-\Delta + \zeta)\mathbf{u} + \nabla p &= \mathbf{h} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.1}$$

It is assumed that the number ζ is not the eigenvalue of the Stokes operator L. We denote L_ζ by the Stokes operator with the parameter ζ as follows:

$$\mathrm{L}_\zeta[\mathbf{u}, p] := \begin{cases} (-\Delta + \zeta)\mathbf{u} + \nabla p, \\ \operatorname{div} \mathbf{u}. \end{cases} \tag{2.2}$$

We next give an existence result of (2.1) and its a priori estimate.

Lemma 2.1. *Let* $\xi := \mathcal{R}\mathrm{e}\,\zeta > 0$. (i) *For* $\mathbf{h} \in \mathrm{V}^{-1}$, *there is a unique solution* $[\mathbf{u}, p] = \mathrm{L}_\zeta^{-1}[\mathbf{h}, 0] \in \mathrm{V} \times \mathrm{L}_0^2$ *of* (2.1), *satisfying the a priori estimate*

$$\big(1 + |\zeta|\big)^{1/2}\|\nabla \mathbf{u}\|_0 + \big(1 + |\zeta|\big)\|\mathbf{u}\|_0 + \|p\|_0 \leqslant C\big(1 + |\zeta|\big)^{1/2}\|\mathbf{h}\|_{\mathrm{V}^{-1}}, \tag{2.3}$$

where C *is a constant not depending on* ζ. (ii) *If* $\mathbf{h} \in \mathbf{L}^2$, *then*

$$\big(1 + |\zeta|\big)^{1/2}\|\nabla \mathbf{u}\|_0 + \big(1 + |\zeta|\big)\|\mathbf{u}\|_0 + \|p\|_0 \leqslant C\|\mathbf{h}\|_0,$$

where C is a constant not depending on ζ. (iii) *If* $\mathrm{V}^{-s} := [\mathrm{H}, \mathrm{V}^{-1}]_s$ *is the intermediate space of* H *and* V^{-1} *for* $0 < s < 1$, *then the velocity* $\mathbf{u}$ *satisfies*

$$|\zeta|\|\mathbf{u}\|_{\mathrm{V}^{-s}} \leqslant C\|\mathbf{h}\|_{\mathrm{V}^{-s}}$$

for a constant C.

Proof. (i) Multiplying the first equation of (2.1) by $\zeta\mathbf{u}$ and the second equation by $\bar{\zeta}p$, adding together and doing integration by parts,

$$\bar{\zeta}\|\nabla\mathbf{u}\|_0^2 + |\zeta|^2\|\mathbf{u}\|_0^2 + 2i\Im\left\{\int_\Omega \bar{p}\,\mathrm{div}(\zeta\mathbf{u})\,\mathrm{d}\mathbf{x}\right\} = \bar{\zeta}\int_\Omega \mathbf{h}\cdot\bar{\mathbf{u}}\,\mathrm{d}\mathbf{x}, \tag{2.4}$$

where the "bar" means the complex conjugate. Taking the real part of (2.4),

$$\xi\|\nabla\mathbf{u}\|_0^2 + |\zeta|^2\|\mathbf{u}\|_0^2 \leqslant C|\zeta|\int_\Omega |\mathbf{h}\cdot\bar{\mathbf{u}}|\,\mathrm{d}\mathbf{x}, \tag{2.5}$$

where C is a constant not depending on $\zeta := \xi + i\eta$. Next, multiplying the first equation of (2.1) by $(\eta + i\xi)\mathbf{u}$ and the second equation by $(\eta - i\xi)p$ and adding together,

$$(\eta - i\xi)\|\nabla\mathbf{u}\|_0^2 + (\xi + i\eta)(\eta - i\xi)\|\mathbf{u}\|_0^2 + 2i\Im\left\{\int_\Omega \bar{p}\,\mathrm{div}\big[(\eta + i\xi)\mathbf{u}\big]\,\mathrm{d}\mathbf{x}\right\}$$
$$= (\eta - i\xi)\int_\Omega \mathbf{h}\cdot\bar{\mathbf{u}}\,\mathrm{d}\mathbf{x}.$$

Taking the real part to both sides of above equation,

$$|\eta|\|\nabla\mathbf{u}\|_0^2 + 2\xi|\eta|\|\mathbf{u}\|_0^2 \leqslant |\zeta|\int_\Omega |\mathbf{h}\cdot\bar{\mathbf{u}}|\,\mathrm{d}\mathbf{x}.$$

Combining this with (2.5),

$$|\zeta|\|\nabla\mathbf{u}\|_0^2 + |\zeta|^2\|\mathbf{u}\|_0^2 \leqslant C|\zeta|\int_\Omega |\mathbf{h}\cdot\bar{\mathbf{u}}|\,\mathrm{d}\mathbf{x}, \tag{2.6}$$

where C is a generic constant not depending on ζ. Since

$$|\zeta|\int_\Omega |\mathbf{h}\cdot\bar{\mathbf{u}}|\,\mathrm{d}\mathbf{x} \leqslant \epsilon|\zeta|^2\|\mathbf{u}\|_0^2 + C\epsilon^{-1}\|\mathbf{h}\|_0^2, \quad \forall\epsilon > 0,$$

and taking $\epsilon = 1/2$, combining with (2.6),

$$|\zeta|^{1/2}\|\nabla\mathbf{u}\|_0 + |\zeta|\|\mathbf{u}\|_0 \leqslant C\|\mathbf{h}\|_0. \tag{2.7}$$

On the other hand, from (2.6),

$$\begin{aligned}\|\nabla\mathbf{u}\|_0^2+|\zeta|\|\mathbf{u}\|_0^2&\leqslant C\|\mathbf{h}\|_{\mathrm{V}^{-1}}\|\mathbf{u}\|_1\\&\leqslant C\|\mathbf{h}\|_{\mathrm{V}^{-1}}\big(\|\mathbf{u}\|_0+\|\nabla\mathbf{u}\|_0\big)\\&\leqslant \delta_1\|\mathbf{u}\|_0^2+\delta_2\|\nabla\mathbf{u}\|_0^2+C\delta_1^{-1}\|\mathbf{h}\|_{\mathrm{V}^{-1}}^2+C\delta_2^{-1}\|\mathbf{h}\|_{\mathrm{V}^{-1}}^2,\end{aligned}$$

where δ_i are arbitrary numbers. Taking $\delta_1=|\zeta|/2$ and $\delta_2=1/2$, we have

$$|\zeta|^{1/2}\|\nabla\mathbf{u}\|_0+|\zeta|\|\mathbf{u}\|_0\leqslant C\big(1+|\zeta|\big)^{1/2}\|\mathbf{h}\|_{\mathrm{V}^{-1}},$$

where C is a constant.

To estimate the pressure, recalling the space $\mathrm{V}=\{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\colon\ \operatorname{div}\mathbf{v}=0\}$, a weak form of problem (2.1) is to find $\mathbf{u}\in\mathrm{V}$ such that

$$\int_\Omega(-\Delta\mathbf{u}+\zeta\mathbf{u}-\mathbf{h})\cdot\bar{\mathbf{v}}\,d\mathbf{x}=0,\quad\forall\mathbf{v}\in\mathrm{V}.\tag{2.8}$$

By Temam [31, Proposition 1.1.1], there exists a function $p\in\mathrm{L}^2$ such that $\nabla p=\Delta\mathbf{u}-\zeta\mathbf{u}+\mathbf{h}$ holds in the distributional sense. So

$$\|p\|_0\leqslant C\big(\|\nabla\mathbf{u}\|_0+|\zeta|\|\mathbf{u}\|_0+\|\mathbf{h}\|_{\mathrm{V}^{-1}}\big)\tag{2.9}$$

for a constant C. Thus (2.3) follows.

(ii) Using the inequalities (2.7) and (2.9), the required inequality follows.

(iii) From (2.8) one has $\|\nabla\mathbf{u}\|_0\leqslant C\|\mathbf{h}\|_{\mathrm{V}^{-1}}$ for a constant C. Since

$$|\zeta|\int_\Omega\frac{|\mathbf{u}\cdot\bar{\mathbf{v}}|\,d\mathbf{x}}{\|\mathbf{v}\|_1}\leqslant\|\Delta\mathbf{u}+\mathbf{h}\|_{\mathrm{V}^{-1}},\quad\forall\mathbf{v}\in\mathrm{V},\ \mathbf{v}\neq0,$$

we have

$$\begin{aligned}|\zeta|\|\mathbf{u}\|_{\mathrm{V}^{-1}}&\leqslant\|\Delta\mathbf{u}\|_{\mathrm{V}^{-1}}+\|\mathbf{h}\|_{\mathrm{V}^{-1}}\\&\leqslant\|\Delta\mathbf{u}\|_{-1}+\|\mathbf{h}\|_{\mathrm{V}^{-1}}\\&\leqslant C\big(\|\nabla\mathbf{u}\|_0+\|\mathbf{h}\|_{\mathrm{V}^{-1}}\big)\\&\leqslant C\|\mathbf{h}\|_{\mathrm{V}^{-1}}\end{aligned}\tag{2.10}$$

where (2.3) was used in the above last inequality. Thus $|\zeta|\|\mathbf{u}\|_{\mathrm{V}^{-1}}\leqslant C\|\mathbf{h}\|_{\mathrm{V}^{-1}}$ for a constant C. Next we estimate the function $\mathbf{u}$ in the space V^{-s} $(0<s<1)$ (see Lions and Magenes [23]). Recall that

$$\mathrm{V}\subset\mathrm{H}\subset\mathbf{L}^2,\quad\mathrm{V}\subset\mathbf{H}_0^1=\mathrm{V}\oplus\mathrm{V}^\perp\subset\mathrm{H}\oplus\mathrm{H}^\perp=\mathbf{L}^2.$$

So $\mathrm{H}\subset\mathrm{H}^{-1}\subset\mathrm{V}^{-1}$ and $\mathrm{V}^{-s}\subset\mathrm{V}^{-1}$ for $0<s<1$. Using (2.10) and $|\zeta|\|\mathbf{u}\|_0\leqslant C\|\mathbf{h}\|_0$ by (2.7), the required inequality follows. □

Next we are going to construct the coefficients of the singular functions of the solution of (2.1) in an explicit way. We use similar procedures like the ones used for the Laplace problem with parameter (see [11]) but much more complicated computational techniques are involved because we have to handle the Stokes operator with parameter.

Using the Stokes operator L in (1.12), equations in (2.1) can be rewritten in the form $\mathrm{L}[\mathbf{u}, p] = [\mathbf{h} - \zeta\mathbf{u}, 0]$. By Theorem 2.1, the solution $[\mathbf{u}, p]$ of (2.1) belongs to the space

$$\big(\mathbf{H}^s \times \mathrm{H}^{s-1}\big) \oplus \operatorname{span}\big\{[\Phi_i, \phi_i]\colon\ i = 1, \ldots, N\big\},$$

where $s_N < s \leqslant 2$ and the number $N = 1$ or 2, depending on the angle $\omega > \pi$ (see (1.5)–(1.6)). Since $e^{-r\sqrt{\zeta}} \to 1$ as $r \to 0$, we have

$$\big(1 - e^{-r\sqrt{\zeta}}\big)[\Phi_i, \phi_i] \in \mathbf{H}^{s_i} \times \mathrm{H}^{s_i - 1},$$

so $[\Phi_i, \phi_i]$ and $e^{-r\sqrt{\zeta}}[\Phi_i, \phi_i]$ have same behaviors near $r = 0$. Also the solution $[\mathbf{u}, p]$ belongs to the space

$$[\mathbf{u}, p] \in \big(\mathbf{H}^s \times \mathrm{H}^{s-1}\big) \oplus \operatorname{span}\big\{e^{-r\sqrt{\zeta}}[\Phi_i, \phi_i]\colon\ i = 1, \ldots, N\big\},$$

where $s_N < s \leqslant 2$. This implies that there exist constants c_i, $1 \leqslant i \leqslant N$, depending on ζ such that

$$[\mathbf{u}, p] = \sum_{i=1}^{N} c_i e^{-r\sqrt{\zeta}}[\Phi_i, \phi_i] + [\mathbf{u}_R, p_R], \tag{2.11}$$

where $[\mathbf{u}_R, p_R] \in \mathbf{H}^s \times \mathrm{H}^{s-1}$ for $s_N < s \leqslant 2$. Hence $e^{-r\sqrt{\zeta}}[\Phi_i, \phi_i]$ is the singularity pair of velocity and pressure depending on the parameter ζ. Since the coefficient c_i of the singular part in (2.11) depends on $\mathbf{h}$, we define a mapping Λ_i^ζ by

$$\Lambda_i^\zeta(\mathbf{h}, g) := c_i,$$

where we set $g = 0$.

In general, if the mapping Λ_i^ζ is a continuous linear functional on $\mathrm{V}^{s-2} \times \mathrm{H}^{s-1}$ for $s > s_i$, then there exists $[\mathbf{v}_i, \eta_i] \in \mathrm{V}^{2-s} \times \mathrm{H}^{1-s}$ such that

$$\Lambda_i^\zeta(\mathbf{h}, g) = \int_\Omega \mathbf{h} \cdot \bar{\mathbf{v}}_i + g\bar{\eta}_i \,\mathrm{d}\mathbf{x}. \tag{2.12}$$

Using the eigenfunction pair $[\mathcal{T}_i, \xi_i]$ corresponding to the eigenvalue λ_i of the Stokes operator L (see (1.4)–(1.6)) we will define the pair $[\mathbf{v}_i, \eta_i]$ of (2.12) having the form

$$\mathbf{v}_i(\zeta) = \gamma_i\big(e^{-r\sqrt{\zeta}} r^{-\lambda_i}\mathcal{T}_i + \Psi_i\big), \qquad \eta_i(\zeta) = \gamma_i\big(e^{-r\sqrt{\zeta}} r^{-\lambda_i - 1}\xi_i + \psi_i\big). \tag{2.13}$$

In subsequent lemmas we will construct the pair $[\Psi_i, \psi_i]$ and the number γ_i in (2.13).

Lemma 2.2. *The pair* $[\mathbf{v}_i, \eta_i]$ *in* (2.13) *has the following properties*

(i) $[\mathbf{v}_i, \eta_i] \in \mathrm{V}^{2-s} \times \mathrm{H}^{1-s}$ *for* $s > s_i$,
(ii) $\mathrm{L}_\zeta[\mathbf{v}_i, \eta_i] = 0$,
(iii) $\mathbf{v}_i = 0$ *on* Γ, *except the corners of* Ω,
(iv) $\int_\Omega \mathrm{L}_\zeta[\Phi_i, \phi_i] \cdot [\bar{\mathbf{v}}_i, \bar{\eta}_i]\,\mathrm{d}\mathbf{x} = 1$.

Proof. Set $[\mathbf{u}_1, p_1] = [\mathbf{u}, p]$ and write, for $j = 1, \dots, N$,

$$[\mathbf{u}_j, p_j] = \sum_{i=1}^{j} c_i e^{-r\sqrt{\zeta}}[\Phi_i, \phi_i] + [\mathbf{u}_{j+1}, p_{j+1}].$$

Inserting $[\mathbf{u}_j, p_j]$ into the equation $\mathrm{L}_\zeta[\mathbf{u}, p] = [\mathbf{h}, g]$, multiplying by $[\mathbf{v}_j, \eta_j]$, and integrating over Ω, we have, for $j = 1, \dots, N$,

$$\begin{aligned} 0 &= \int_\Omega \mathrm{L}_\zeta[\mathbf{u}_{j+1}, p_{j+1}] \cdot [\bar{\mathbf{v}}_j, \bar{\eta}_j]\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega [\mathbf{u}_{j+1}, p_{j+1}] \cdot \mathrm{L}_\zeta[\bar{\mathbf{v}}_j, \bar{\eta}_j]\,\mathrm{d}\mathbf{x}. \end{aligned}$$

Since $[\mathbf{u}_{j+1}, p_{j+1}]$ is a smoother part of the solution of (2.1) for any function $\mathbf{h} \in \mathrm{V}^{s-2}$ for $s > s_j$, we conclude that $\mathrm{L}_\zeta[\mathbf{v}_j, \eta_j] = 0$. So (ii) follows. Inserting the identity

$$[\Phi_j, \phi_j] = e^{-r\sqrt{\zeta}}[\Phi_j, \phi_j] + \big(1 - e^{-r\sqrt{\zeta}}\big)[\Phi_j, \phi_j]$$

into the operator L_ζ, multiplying by $[\mathbf{v}_j, \eta_j]$, and integrating over Ω and recalling that $(1 - e^{-r\sqrt{\zeta}})[\Phi_j, \phi_j] \in \mathbf{H}^s \times \mathrm{H}^{s-1}$, the identity (iv) follows. (iii) follows from the boundary condition of (2.19) given below. (i) will be shown later (see Lemma 2.4 below). □

For $i = 1, \dots, N$, we set

$$E_i = e^{-r\sqrt{\zeta}} r^{-\lambda_i} \mathcal{T}_i(\theta), \qquad \sigma_i = e^{-r\sqrt{\zeta}} r^{-\lambda_i - 1} \xi_i(\theta). \tag{2.14}$$

Lemma 2.3. *For* $1 \leqslant i \leqslant N$ *let* $[\mathbf{F}_i, g_i] := \mathrm{L}_\zeta[E_i, \sigma_i]$ *the image of* $[E_i, \sigma_i]$ *by the operator* L_ζ. *If* $s > s_i$, *then* $[\mathbf{F}_i, g_i] \in \mathbf{H}^{-s} \times \mathrm{H}^{1-s}$, *with the inequality*

$$\big\|[\mathbf{F}_i, g_i]\big\|_{\mathbf{H}^{-s} \times \mathrm{H}^{1-s}} \leqslant C\big(1 + |\zeta|\big)^{(s_i - s)/2}.$$

Proof. Let U and V be the polar components of the vector $\mathbf{u} = [u_1, u_2]$ defined by $U = u_1 \cos\theta + u_2 \sin\theta$ and $V = -u_1 \sin\theta + u_2 \cos\theta$. In the polar coordinate (r, θ) the Stokes operator L_ζ has the form

$$\mathrm{L}_\zeta = \begin{pmatrix} -\Delta + \zeta + r^{-2} & 2r^{-2}\partial_\theta & \partial_r \\ -2r^{-2}\partial_\theta & -\Delta + \zeta + r^{-2} & r^{-1}\partial_\theta \\ r^{-1} + \partial_r & r^{-1}\partial_\theta & 0 \end{pmatrix} \tag{2.15}$$

where $\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}$. We write $\mathcal{T}_i = [\mathcal{T}_{i,1}, \mathcal{T}_{i,2}]$, $E_i = [E_{i,1}, E_{i,2}]$ and $\mathbf{F}_i = [F_{i,1}, F_{i,2}]$. Then $[\mathbf{F}_i, g_i]$ has the following three components

$$\begin{aligned} F_{i,1} &= (1-2\lambda_i)\sqrt{\zeta}r^{-1}E_{i,1} - \sqrt{\zeta}\sigma_i, \\ F_{i,2} &= (1-2\lambda_i)\sqrt{\zeta}r^{-1}E_{i,2}, \\ g_i &= -\sqrt{\zeta}e^{-r\sqrt{\zeta}}r^{-\lambda_i}\mathcal{T}_{i,1}, \end{aligned} \tag{2.16}$$

which follows from the fact (see [19, (5.1.4)]) that: for $\lambda = -\lambda_i$, the eigenpair $[\mathcal{T}_i, \xi_i]$ satisfies

$$\begin{aligned} -\mathcal{T}_{i,1}'' + \left(1-\lambda_i^2\right)\mathcal{T}_{i,1} + 2\mathcal{T}_{i,2}' &= (-\lambda_i - 1)\xi_i, \\ -\mathcal{T}_{i,2}'' + \left(1-\lambda_i^2\right)\mathcal{T}_{i,2} - 2\mathcal{T}_{i,1}' &= -\xi_i', \\ (1-\lambda_i)\mathcal{T}_{i,1} + \mathcal{T}_{i,2}' &= 0. \end{aligned} \tag{2.17}$$

Note that equations in (2.17) can be obtained by applying the pair

$$\left[r^{-\lambda_i}\mathcal{T}_i(\theta), r^{-\lambda_i-1}\xi_i(\theta)\right]$$

to the Stokes operator transformed by the polar coordinate [19, Section 5.1].

Setting $\delta_i = \mathcal{R}\mathrm{e}\,\lambda_i$, the function $\mathbf{F}_i := (F_{i1}, F_{i2})$ of (2.16) is estimated by

$$|\mathbf{F}_i| \leqslant C\left(1+|\zeta|^{1/2}\right)r^{-\delta_i-1}e^{-r\sqrt{|\zeta|}\cos(\theta_*/2)},$$

where $\theta_* \in (-\pi/2, \pi/2)$ is the argument of ζ with $\mathcal{R}\mathrm{e}\,\zeta > 0$. Since

$$\sqrt{2}/2 \leqslant \cos(\theta_*/2) \leqslant 1, \quad \theta_* \in (-\pi/2, \pi/2),$$

and using [10, Theorem 1.4.4.4],

$$\begin{aligned} \|\mathbf{F}_i\|_{-s}^2 &\leqslant C\left\|r^s\mathbf{F}_i\right\|_{0,\Omega}^2 \\ &\leqslant C\left(1+|\zeta|\right)\int_0^\infty r^{2(s-\delta_{i-1})+1}e^{-\sqrt{2}r\sqrt{|\zeta|}}\,\mathrm{d}r \\ &\qquad \left(\text{letting } t = r\sqrt{1+|\zeta|} \text{ and } c_1 = \sqrt{|\zeta|}/\sqrt{1+|\zeta|}\right) \\ &\leqslant C\left(1+|\zeta|\right)^{-s+\delta_i+1}\int_0^\infty t^{2(s-\delta_i)-3}e^{-2c_1t}\,\mathrm{d}t \\ &\leqslant C\left(1+|\zeta|\right)^{s_i-s}, \end{aligned}$$

which follows from the fact that

$$\int_0^\infty t^{2(s-\delta_i)-3}e^{-2c_1t}\,\mathrm{d}t < \infty, \quad \text{if } s > s_i.$$

Thus $\|\mathbf{F}_i\|_{1-s,\Omega} \leqslant C(1+|\zeta|)^{(s_i-s)/2}$ for a constant C. Likewise, the function g_i of (2.11) is estimated by

$$|g_i| \leqslant C\big(1+|\zeta|^{1/2}\big)r^{-\delta_i}e^{-r\sqrt{|\zeta|}\cos(\theta_*/2)}$$

where θ_* is the argument of ζ with $\mathcal{R}\mathrm{e}\,\zeta > 0$. Hence, if $s > s_i$, one can show $\|g_i\|_{1-s,\Omega} \leqslant C(1+|\zeta|)^{(s_i-s)/2}$ for a constant C. $\square$

In next lemma we show the first property (i) of Lemma 2.3.

Lemma 2.4. *For each $1 \leqslant i \leqslant N$, there is a constant K, not depending on ζ, such that the pair $[\mathbf{v}_i, \eta_i]$ of* (2.13) *satisfies*

$$\|\mathbf{v}_i\|_{2-s} + \|\eta_i\|_{1-s} \leqslant K\big(1+|\zeta|\big)^{(s_i-s)/2} \quad \textit{for } s > s_i. \tag{2.18}$$

Proof. Recall that $[\mathbf{F}_i, g_i] := \mathrm{L}_\zeta[E_i, \sigma_i]$. For $s_i < s < s_{i+1}$, $[\mathbf{F}_i, g_i] \in \mathbf{H}^{-s} \times \mathrm{H}^{1-s}$ by Lemma 2.3. From the identity (ii) in Lemma 2.2, one can look for the pair $[\Psi_i, \psi_i] \in \mathbf{H}^{2-s} \times \mathrm{H}^{1-s}$, satisfying

$$\mathrm{L}_\zeta[\Psi_i, \psi_i] = -[\mathbf{F}_i, g_i] \quad \text{in } \Omega, \quad \Psi_i|_\Gamma = -E_i|_\Gamma. \tag{2.19}$$

Using the ellipticity of L_ζ and Lemma 2.3, one has

$$\begin{aligned}\|\Psi_i\|_{2-s} + \|\psi_i\|_{1-s} &\leqslant C\big(\|\mathbf{F}_i\|_{-s} + \|g_i\|_{1-s}\big)\\ &\leqslant C\big(1+|\zeta|\big)^{(s_i-s)/2}\end{aligned} \tag{2.20}$$

where C is a generic constant.

We next compute $\|E_i\|_{2-s}$ and $\|\sigma_i\|_{1-s}$ as follows. Note that $1-\mathcal{R}\mathrm{e}\,\lambda_{i+1} < 2-s < 1-\mathcal{R}\mathrm{e}\,\lambda_i$ and $-\mathcal{R}\mathrm{e}\,\lambda_{i+1} < 1-s < -\mathcal{R}\mathrm{e}\,\lambda_i < 0$. Using [10, Theorem 1.4.4.4], we have

$$\begin{aligned}\|\sigma_i\|_{1-s}^2 &\leqslant \big\|r^{s-1}\sigma_i\big\|_0^2\\ &\leqslant C\int_0^\infty r^{2(s-\mathcal{R}\mathrm{e}\,\lambda_i-1)-1}e^{-r\sqrt{2|\zeta|}}\,\mathrm{d}r\\ &\leqslant C\big(1+|\zeta|\big)^{s_i-s},\end{aligned} \tag{2.21}$$

where C is a constant. To show that $\|E_i\|_{2-s} < \infty$, it suffices to show that the following quantities

$$\int_0^1\int_0^1 \frac{|e^{-r\sqrt{\zeta}}r^{-\lambda_i} - e^{-r_1\sqrt{\zeta}}r_1^{-\lambda_i}|^2}{|r-r_1|^{1+2(2-s)}}r\,\mathrm{d}r\,\mathrm{d}r_1\int_{\omega_1}^{\omega_2}\big|\mathcal{T}_i(\theta)\big|^2\,\mathrm{d}\theta, \tag{2.22}$$

$$\int_{\omega_1}^{\omega_2}\int_{\omega_1}^{\omega_2}\frac{|\mathcal{T}_i(\theta) - \mathcal{T}_i(\theta_1)|^2}{|\theta-\theta_1|^{1+2(2-s)}}\,\mathrm{d}\theta\,\mathrm{d}\theta_1\int_0^1\big|e^{-r\sqrt{\zeta}}r^{-\lambda_i}\big|^2r^{1-2(2-s)}\,\mathrm{d}r \tag{2.23}$$

are estimated by $C(1+|\zeta|)^{s_i-s}$ for a constant C. Recall that $\cos(\theta_*/2) \geqslant 1/\sqrt{2}$ where θ_* is the argument of ζ with $\mathcal{R}\mathrm{e}\,\zeta > 0$. Since $s > s_i = \mathcal{R}\mathrm{e}\,\lambda_i + 1$ and $\mathcal{T}_i$ is a smooth function of θ, the term (2.23) is estimated by

$$\int_0^1 \left|e^{-r\sqrt{\zeta}} r^{-\lambda_i}\right|^2 r^{2s-3}\,\mathrm{d}r = \int_0^1 e^{-2r\sqrt{|\zeta|}\cos(\theta_*/2)} r^{2(s-\delta_i-1)-1}\,\mathrm{d}r \leqslant C\big(1+|\zeta|\big)^{s_i-s},$$

where $\delta_i := \mathcal{R}\mathrm{e}\,\lambda_i$. We next estimate (2.22). Note that

$$\begin{aligned}\left|e^{-r\sqrt{\zeta}} r^{-\lambda_i} - e^{-r_1\sqrt{\zeta}} r_1^{-\lambda_i}\right| &\leqslant r^{-\delta_i}\left|e^{-r\sqrt{\zeta}} - e^{-r_1\sqrt{\zeta}}\right| + e^{-r_1\sqrt{\zeta}}\left|r^{-\lambda_i} - r_1^{-\lambda_i}\right| \\ &\leqslant C\big(r^{-\delta_i}\left|e^{-ar\sqrt{|\zeta|}} - e^{-ar_1\sqrt{|\zeta|}}\right| + e^{-ar_1\sqrt{|\zeta|}}\left|r^{-\lambda_i} - r_1^{-\lambda_i}\right|\big),\end{aligned}$$

where $a = \cos(\theta_*/2)$. Letting $\eta = a\sqrt{1+|\zeta|}$, $\tau = \eta r$, $\tau_1 = \eta r_1$ and $c = \sqrt{|\zeta|}/\eta$, we have

$$\begin{aligned}&\int_0^1\int_0^1 \frac{|e^{-ar\sqrt{|\zeta|}} - e^{-ar_1\sqrt{|\zeta|}}|^2}{|r-r_1|^{5-2s}} r^{1-2\lambda_i}\,\mathrm{d}r\,\mathrm{d}r_1 \\ &\quad = \big(a^2\big(1+|\zeta|\big)\big)^{s_i-s}\int_0^\eta\int_0^\eta \frac{|e^{-c\tau} - e^{-c\tau_1}|^2}{|\tau-\tau_1|^{5-2s}}\,\mathrm{d}\tau\,\mathrm{d}\tau_1 \\ &\quad \leqslant C\big(1+|\zeta|\big)^{s_i-s}\int_0^\eta\int_0^\eta e^{-c\tau_*}|\tau-\tau_1|^{2s-3}\,\mathrm{d}\tau\,\mathrm{d}\tau_1 \\ &\quad \leqslant C\big(1+|\zeta|\big)^{s_i-s},\end{aligned} \tag{2.24}$$

where τ_* is a number between τ and τ_1. Similarly, for $\eta = a\sqrt{1+|\zeta|}$, we have

$$\begin{aligned}&\int_0^1\int_0^1 \frac{|r^{-\lambda_i} - r_1^{-\lambda_i}|^2}{|r-r_1|^{5-2s}} e^{-2ar_1\sqrt{|\zeta|}} r\,\mathrm{d}r\,\mathrm{d}r_1 \\ &\quad = \big(a^2\big(1+|\zeta|\big)\big)^{s_i-s}\int_0^\eta\int_0^\eta \frac{|\tau^{-\lambda_i} - \tau_1^{-\lambda_i}|^2}{|\tau-\tau_1|^{5-2s}} \tau e^{-2c\tau_1}\,\mathrm{d}\tau\,\mathrm{d}\tau_1 \\ &\quad \leqslant C\big(1+|\zeta|\big)^{s_i-s}\int_0^\eta\int_0^\eta \tau e^{-2c\tau_1}|\tau\tau_1|^{-2\delta_i}|\tau-\tau_1|^{2s-5}\left|\tau^{\lambda_i} - \tau_1^{\lambda_i}\right|^2\mathrm{d}\tau\,\mathrm{d}\tau_1 \quad (\text{setting } \tau = \tau_1 t) \\ &\quad \leqslant C\big(1+|\zeta|\big)^{s_i-s}\int_0^\eta \tau_1^{2s-2\delta_i-3}e^{-2c\tau_1}\,\mathrm{d}\tau_1\int_0^{\eta/\tau_1} t^{1-2\delta_i}|t-1|^{2s+2\delta_i-5}\,\mathrm{d}t \\ &\quad \leqslant C\big(1+|\zeta|\big)^{s_i-s},\end{aligned} \tag{2.25}$$

where the last inequality follows from the fact that

$$2s - 2\delta_i - 3 > -1, \qquad 2s + 2\delta_i - 5 > -1, \qquad 1 - 2\delta_i > -1.$$

Thus $\|E_i\|_{2-s} \leqslant C(1+|\zeta|)^{(s_i-s)/2}$ for a constant C.

By the identity (iv) in Lemma 2.2 we compute the number γ_i of (2.13). This gives

$$1/\gamma_i = \int_\Omega \mathrm{L}_\zeta[\Phi_i, \phi_i] \cdot \big([\bar{E}_i, \bar{\sigma}_i] + [\bar{\Psi}_i, \bar{\psi}_i]\big)\,\mathrm{d}\mathbf{x}. \tag{2.26}$$

Using (2.19),

$$\begin{aligned}
\int_\Omega \mathrm{L}_\zeta[\Phi_i, \phi_i] \cdot [\bar{\Psi}_i, \bar{\psi}_i]\,\mathrm{d}\mathbf{x} &= \int_\Omega \Phi_i(\zeta\bar{\Psi}_i - \Delta\bar{\Psi}_i + \nabla\bar{\psi}_i)\,\mathrm{d}\mathbf{x} \\
&\quad - \int_\Omega \phi_i \operatorname{div}\bar{\Psi}_i\,\mathrm{d}\mathbf{x} + \int_\Gamma \left(\phi_i\mathbf{n} - \frac{\partial\Phi_i}{\partial\mathbf{n}}\right)\bar{\Psi}_i + \Phi_i\left(\frac{\partial\bar{\Psi}_i}{\partial\mathbf{n}} - \bar{\psi}_i\mathbf{n}\right)\mathrm{d}s \\
&= \int_\Omega [\Phi_i, \phi_i] \cdot \mathrm{L}_\zeta[\bar{\Psi}_i, \bar{\psi}_i]\,\mathrm{d}\mathbf{x} \\
&= -\int_\Omega [\Phi_i, \phi_i] \cdot \mathrm{L}_\zeta[\bar{E}_i, \bar{\sigma}_i]\,\mathrm{d}\mathbf{x},
\end{aligned} \tag{2.27}$$

where we used Eq. (2.19) in the last integral. Combining (2.26) and (2.27),

$$\begin{aligned}
1/\gamma_i &= \int_\Omega \mathrm{L}_\zeta[\Phi_i, \phi_i] \cdot [\bar{E}_i, \bar{\sigma}_i] - [\Phi_i, \phi_i] \cdot \mathrm{L}_\zeta[\bar{E}_i, \bar{\sigma}_i]\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega -\Delta\Phi_i\,\bar{E}_i + \Phi_i\,\Delta\bar{E}_i\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla\phi_i \cdot \bar{E}_i + \phi_i \operatorname{div}\bar{E}_i\,\mathrm{d}\mathbf{x} - \int_\Omega \Phi_i \cdot \nabla\bar{\sigma}_i + \bar{\sigma}_i \operatorname{div}\Phi_i\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega -\Delta\Phi_i\,\bar{E}_i + \Phi_i\,\Delta\bar{E}_i\,\mathrm{d}\mathbf{x} + \lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon} \phi_i\bar{E}_i \cdot \mathbf{n} - \bar{\sigma}_i\Phi_i \cdot \mathbf{n}\,\mathrm{d}s \\
&= \int_\Omega -\bar{E}_i\,\Delta\Phi_i + \Phi_i\,\Delta\bar{E}_i\,\mathrm{d}\mathbf{x},
\end{aligned} \tag{2.28}$$

which follows from

$$\lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon} \phi_i\bar{E}_i \cdot \mathbf{n} - \bar{\sigma}_i\Phi_i \cdot \mathbf{n}\,\mathrm{d}s = \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} \phi_i\bar{E}_i \cdot \mathbf{n} - \bar{\sigma}_i\Phi_i \cdot \mathbf{n}\,\mathrm{d}s = 0,$$

where $\Omega_\epsilon = \Omega \cap \{r > \epsilon\}$ and Γ_ϵ is the arc defined by $r = \epsilon$ and $\omega_1 < \theta < \omega_2$ oriented positively with respect to 0. Note that the last integral (2.28) may not vanish because the classical Green formula cannot be applicable [11,12]. To handle this, we calculate this integral as a limit as $\epsilon \to 0$ of the corresponding integral over Ω_ϵ. Now

$$\begin{aligned} 1/\gamma_i &= \lim_{\epsilon\to 0} \int_{\Omega_\epsilon} -\bar{E}_i \Delta \Phi_i + \Phi_i \Delta \bar{E}_i \, \mathrm{d}\mathbf{x} \\ &= \lim_{\epsilon\to 0} \int_{\partial\Omega_\epsilon} -\bar{E}_i \frac{\partial \Phi_i}{\partial \mathbf{n}} + \Phi_i \frac{\partial \bar{E}_i}{\partial \mathbf{n}} \, \mathrm{d}s \\ &= \lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} -\bar{E}_i \frac{\partial \Phi_i}{\partial \mathbf{n}} + \Phi_i \frac{\partial \bar{E}_i}{\partial \mathbf{n}} \, \mathrm{d}s. \end{aligned} \tag{2.29}$$

So

$$\begin{aligned} 1/\gamma_i &= \lim_{\epsilon\to 0} \epsilon \int_{\omega_1}^{\omega_2} -\bar{E}_i \frac{\partial \Phi_i}{\partial r} + \Phi_i \frac{\partial \bar{E}_i}{\partial r} \, \mathrm{d}\theta \\ &= \lim_{\epsilon\to 0} e^{-\epsilon\zeta_1}(-2\lambda_i - \epsilon\zeta_1) \int_{\omega_1}^{\omega_2} \left|\mathcal{T}_i(\theta)\right|^2 \mathrm{d}\theta \\ &= -2\lambda_i \int_{\omega_1}^{\omega_2} \left|\mathcal{T}_i(\theta)\right|^2 \mathrm{d}\theta \end{aligned} \tag{2.30}$$

where ζ_1 is the conjugate of $\sqrt{\zeta}$. Hence the γ_i is defined by the equality (2.30) since $\mathcal{T}_i$ is a known trigonometric vector function of θ. Thus the inequality (2.18) follows by (2.20)–(2.25). □

Using Theorem 2.1 and Lemmas 2.1–2.4 we show that the solution of (2.1) can be decomposed into singular and regular parts and that the regular part can have an increased regularity. This can be compared with the one of the stationary Stokes problem given in Theorem 2.1.

Theorem 2.2. *Let $\xi := \mathcal{R}\mathrm{e}\,\zeta > 0$. For each $i = 1, \dots, N$ the linear functional Λ_i^ζ, defined by* (2.12)*, is continuous on* $\mathrm{V}^{s-2} \times \mathrm{H}^{s-1}$ *for $s > s_i$ and satisfies the inequality*

$$\left(1 + |\zeta|\right)^{(s-s_i)/2} \left|\Lambda_i^\zeta(\mathbf{h}, 0)\right| \leqslant C\|\mathbf{h}\|_{s-2}, \tag{2.31}$$

and the pair $[\Phi_i(\zeta), \phi_i(\zeta)] \notin \mathbf{H}^{s_i} \times \mathrm{H}^{s_i-1}$, where

$$\Phi_i(\zeta) = \chi e^{-r\sqrt{\zeta}} r^{\lambda_i} \mathcal{T}_i(\theta), \qquad \phi_i(\zeta) = \chi e^{-r\sqrt{\zeta}} r^{\lambda_i - 1} \xi_i(\theta). \tag{2.32}$$

On the other hand, if $s_N < s \leqslant 2$ and $\mathbf{h} \in \mathrm{V}^{s-2}$, then the solution $[\mathbf{u}, p]$ *of* (2.1) *can be split as follows*:

$$[\mathbf{u}, p] = \sum_{i=1}^{N} \Lambda_i^{\zeta}(\mathbf{h}, 0)\big[\Phi_i(\zeta), \phi_i(\zeta)\big] + [\mathbf{u}_R, p_R], \tag{2.33}$$

and the regular part $[\mathbf{u}_R, p_R] \in (\mathbf{H}^s \cap \mathbf{H}_0^1) \times \mathrm{H}^{s-1}$, *and satisfies*

$$\|\mathbf{u}_R\|_s + \|p_R\|_{s-1} + |\zeta|\|\mathbf{u}_R\|_{s-2} \leqslant C\|\mathbf{h}\|_{s-2}. \tag{2.34}$$

Proof. The existence of the functional Λ_i^{ζ} is shown by (2.12), (2.13) and Lemma 2.2. Using Lemma 2.4, we have, for each $i = 1, \dots, N$,

$$\big|\Lambda_i^{\zeta}(\mathbf{h}, 0)\big| \leqslant C\|\mathbf{h}\|_{s-2}\big(\|\mathbf{v}_i\|_{2-s} + \|\eta_i\|_{1-s}\big) \leqslant C\|\mathbf{h}\|_{s-2}\big(1 + |\zeta|\big)^{(s_i-s)/2}$$

for $s > s_i$. Hence (2.31) is shown. Next we show the regularity (2.34). Inserting the decomposition (2.11) with $c_i = \Lambda_i^{\zeta}(\mathbf{h}, 0)$ into the equations in (2.1) and setting

$$\begin{aligned}
\mathbf{h}_s &= \sum_{i=1}^{N} \Lambda_i^{\zeta}(\mathbf{h}, 0)\big[(-\Delta + \zeta)\Phi_i(\zeta) + \nabla\phi_i(\zeta)\big],\\
g_s &= -\sum_{i=1}^{N} \Lambda_i^{\zeta}(\mathbf{h}, 0)\operatorname{div}\Phi_i(\zeta),
\end{aligned} \tag{2.35}$$

the remainder $[\mathbf{u}_R, p_R]$ satisfies

$$\begin{aligned}
(-\Delta + \zeta)\mathbf{u}_R + \nabla p_R &= \mathbf{h} + \mathbf{h}_s \quad \text{in } \Omega,\\
\operatorname{div}\mathbf{u}_R &= g_s \quad \text{in } \Omega,\\
\mathbf{u}_R &= 0 \quad \text{in } \Gamma.
\end{aligned} \tag{2.36}$$

We claim that if $s_N < s < s_{N+1}$, then $\|\mathbf{h}_s\|_{s-2} + \|g_s\|_{s-1} \leqslant C\|\mathbf{h}\|_{s-2}$ for a constant C. For this we first express the functions $\mathbf{h}_s$ and g_s explicitly. We set $\Phi_i = [\Phi_{i,1}, \Phi_{i,2}]$, $\mathcal{T}_i = [\mathcal{T}_{i,1}, \mathcal{T}_{i,2}]$ and

$$\begin{aligned}
k_{i,1} &= \sqrt{\zeta}e^{-r\sqrt{\zeta}}r^{\lambda_i-1}\big[(1+2\lambda_i)\mathcal{T}_{i,1}(\theta) - \phi_i(\theta)\big],\\
k_{i,2} &= (1+2\lambda_i)\sqrt{\zeta}e^{-r\sqrt{\zeta}}r^{\lambda_i-1}\mathcal{T}_{i,2}(\theta),\\
k_{i,3} &= -\sqrt{\zeta}e^{-r\sqrt{\zeta}}r^{\lambda_i}\mathcal{T}_{i,1}(\theta).
\end{aligned} \tag{2.37}$$

Letting $\chi = \chi(r)$, $\mathbf{h}_s = [h_{s,1}, h_{s,2}]$ in (2.35) and using (2.15), we have

$$\begin{aligned}
h_{s,1} &= \sum_{i=1}^{N} \Lambda_i^{\zeta}(\mathbf{h}, 0)\big[\chi k_{i,1} - \Phi_{i,1}(\zeta)\Delta\chi + \phi_i(\zeta)\chi'(r) + 2\chi'(r)\partial_r\Phi_{i,1}(\zeta)\big],\\
h_{s,2} &= \sum_{i=1}^{N} \Lambda_i^{\zeta}(\mathbf{h}, 0)\big[\chi k_{i,2} - \Phi_{i,2}(\zeta)\Delta\chi + 2\chi'(r)\partial_r\Phi_{i,1}(\zeta)\big],\\
g_s &= \sum_{i=1}^{N} \Lambda_i^{\zeta}(\mathbf{h}, 0)\big[\chi k_{i,3} + \Phi_{i,1}(\zeta)\chi'(r)\big].
\end{aligned} \tag{2.38}$$

We first show $\|h_{s,1}\|_{s-2} \leqslant C\|\mathbf{h}\|_{s-2}$ for a constant C. Considering the same procedures used in estimating (2.22) and (2.23), for $i = 1, \dots, N$ and $s_i < s < s_{i+1}$, we have

$$\|k_{i,1}\|_{s-2} + \big\|\partial_r \Phi_{i,1}(\zeta)\big\|_{s-2} \leqslant C\big(1 + |\zeta|\big)^{(s-s_i)/2},$$

and

$$\begin{aligned}\big\|\Phi_{i,1}(\zeta)\Delta\chi - \phi_i(\zeta)\chi'(r)\big\|_{s-2} &\leqslant Ce^{-a\,\mathcal{R}\mathrm{e}\sqrt{\zeta}} \\ &\leqslant C\big(1 + |\zeta|\big)^{(s-s_i)/2}\end{aligned}$$

where $a = \cos(\theta_*/2) > 0$. Hence, using (2.31),

$$\begin{aligned}\|h_{s,1}\|_{s-2} &\leqslant C\|\mathbf{h}\|_{s-2}\sum_{i=1}^{N}\big|\Lambda_i^{\zeta}(\mathbf{h},0)\big|\big(1 + |\zeta|\big)^{(s-s_i)/2} \\ &\leqslant C\|\mathbf{h}\|_{s-2}.\end{aligned} \tag{2.39}$$

Similarly, one can easily show that $\|h_{s,2}\|_{s-2} + \|g_s\|_{s-1} \leqslant C\|\mathbf{h}\|_{s-2}$ for a constant C. Using Theorem 1.1, Lemmas 2.1 and 2.4, the solution $[\mathbf{u}_R, p_R]$ of (2.36) satisfies, for $s_N < s \leqslant 2$,

$$\begin{aligned}\|\mathbf{u}_R\|_s + \|p_R\|_{s-1} &\leqslant C\big(\|\mathbf{h}_s\|_{s-2} + \|g_s\|_{s-1} + |\zeta|\|\mathbf{u}_R\|_{s-2} + \|\mathbf{h}\|_{s-2}\big) \\ &\leqslant C\|\mathbf{h}\|_{s-2}.\end{aligned}$$

Thus we have shown (2.34). □

If $\mathcal{R}\mathrm{e}\,\zeta > 0$, the operator $\mathrm{L}_\zeta = \mathrm{L} + \zeta I$ is invertible from $\mathrm{V}^{-1} \times \mathrm{L}_0^2$ into $\mathrm{V} \times \mathrm{L}_0^2$ by Lemma 2.1. Set $S_\zeta = \mathrm{L}_\zeta^{-1}$ the inverse of L_ζ. By Theorem 2.2 the regular part of S_ζ can be defined as follows:

$$S_{\zeta,R}[\mathbf{h},0] := S_\zeta\big([\mathbf{h},0] + [\mathbf{h}_s, g_s]\big), \tag{2.40}$$

where $[\mathbf{h}_s, g_s]$ is given in (2.35). In other words, if we define $[\mathbf{u}_R, p_R] = S_{\zeta,R}[\mathbf{h},0]$, then $[\mathbf{u}_R, p_R]$ satisfies the inequality (2.34). Since $\mathrm{L}_\zeta = \mathrm{L} + \zeta I$, the resolvent of L is split into singular and regular parts:

$$(\mathrm{L} + \zeta I)^{-1} = S_{\zeta,R} + \sum_{i=1}^{N}\Lambda_i^{\zeta} \otimes \big[\Phi_i^{\zeta}, \phi_i^{\zeta}\big], \tag{2.41}$$

where $S_{\zeta,R}$ is continuous from $\mathrm{V}^{s-2} \times \mathrm{H}^{s-1}$ onto $\mathbf{H}^s \times \mathrm{H}^{s-1}$ for $s \leqslant 2$ and Λ_i^{ζ} is continuous on $\mathrm{V}^{s-2} \times \mathrm{H}^{s-1}$ satisfying $|\Lambda_i^{\zeta}| \leqslant C(1 + |\zeta|)^{(s_i - s)/2}$ for $s > s_i$.

3. The linear problem

We show unique existence of solution for the linearized problem (1.11) and using Theorem 2.2, if the solution is split in a decomposition of singular and regular parts, we show that the coefficient of corner singularity can be expressed in terms of known data **f**, **w** and that the regular part has an increased regularity.

In order to show unique existence of (1.11) we consider the bilinear form

$$a(\mathbf{u},\mathbf{v})=\int_{\Omega}\nabla\mathbf{u}\cdot\nabla\mathbf{v}+(\mathbf{w}\cdot\nabla)\mathbf{u}\mathbf{v}\,d\mathbf{x},\qquad \mathbf{u},\mathbf{v}\in\mathbf{H}_0^1.$$

If $\mathbf{f}\in \mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})$ is given, we find $\mathbf{u}\in\mathrm{L}^2(0,\mathrm{T};\mathrm{V})$ satisfying $\mathbf{u}'\in\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})$ and

$$\begin{aligned}\langle\mathbf{u}',\mathbf{v}\rangle+a(\mathbf{u},\mathbf{v})&=\langle\mathbf{f},\mathbf{v}\rangle,\quad\forall\mathbf{v}\in\mathrm{V},\\ \mathbf{u}(0)&=0,\end{aligned}\tag{3.1}$$

where $\langle\,,\rangle$ is the duality pairing on $\mathrm{V}^{-1}\times\mathrm{V}$.

Lemma 3.1. *Let Ω be a bounded polygon with only one concave vertex. Suppose that $\mathbf{w}\in\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V})$ and $\mathbf{w}(0)=0$.* (i) *If $\mathbf{f}\in\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})$, there is a unique solution $[\mathbf{u},p]$ of* (1.11) *satisfying*

$$\begin{aligned}&\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}+\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})}+\|\mathbf{u}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{L}^2)}+\|p\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2)}\\&\quad\leqslant C\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})},\end{aligned}\tag{3.2}$$

where $C=C(\mathrm{T},\|\mathbf{w}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V})})$. (ii) *If $\mathbf{f}\in\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})$ and $\mathbf{w}'\in\mathrm{L}^2(0,\mathrm{T};\mathrm{V})\cap\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})$, then*

$$\|\mathbf{u}'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}+\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\leqslant C_1\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\tag{3.3}$$

where $C_1=C(\mathrm{T},\|\mathbf{w}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})},\|\mathbf{w}'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})})$. (iii) *Assume that the hypotheses given in* (ii) *hold. If $\mathbf{f}\in\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})$ for $1<s<s_1$ and $\mathbf{w}\in\mathrm{L}^\infty(Q)$, then*

$$\begin{aligned}&\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^s)}+\|p\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{s-1})}+\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}\\&\quad\leqslant C_2\big(\|\mathbf{f}\|_{\mathrm{H}^{s-1}(0,\mathrm{T};\mathrm{V}^{-1})}+\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}\big),\end{aligned}\tag{3.4}$$

where $C_2=C(C_1,\|\mathbf{w}\|_{\infty,Q})$.

Proof. (i) By (3.1) and since $\mathbf{w}\in\mathrm{V}$, we have

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}\|_0^2+2\|\nabla\mathbf{u}\|_0^2&\leqslant 2\|\mathbf{f}\|_{\mathrm{V}^{-1}}\|\mathbf{u}\|_{\mathrm{V}}\\&\leqslant\|\nabla\mathbf{u}\|_0^2+\|\mathbf{u}\|_0^2+\|\mathbf{f}\|_{\mathrm{V}^{-1}}^2.\end{aligned}\tag{3.5}$$

Using the Grönwall's inequality, we get $\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathbf{L}^2)} \leqslant C\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})}$ where $C = C(\mathrm{T})$. Integrating both sides of (3.5), we have $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{L}^2)} + \|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})} \leqslant C\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})}$. Using Temam [31, Lemma 3.4], we have $\|\mathbf{w}\cdot\nabla\mathbf{u}\|_{\mathrm{V}^{-1}} \leqslant \sqrt{2}\|\mathbf{w}\|_{\mathrm{V}}\|\mathbf{u}\|_{\mathrm{V}}$ and from (3.1), one has

$$\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})} \leqslant C\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})}, \tag{3.6}$$

where $C = C(\|\mathbf{w}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V})})$. From Temam [31, Section 3.5], there is a pressure function $p \in \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2)$ satisfying the momentum equation in (1.11). Hence (3.2) follows.

(ii) Differentiating (3.1) with respect to time t and taking $\mathbf{v} = \mathbf{u}'$, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}'\|_0^2 + 2\|\nabla\mathbf{u}'\|_0^2 &\leqslant 2\big|\langle\mathbf{w}'\cdot\nabla\mathbf{u},\mathbf{u}'\rangle\big| + 2\big|\langle\mathbf{f}',\mathbf{u}'\rangle\big| \\ &\leqslant 2\big|\langle\mathbf{w}'\cdot\nabla\mathbf{u},\mathbf{u}'\rangle\big| + 2\|\mathbf{f}'\|_{\mathrm{V}^{-1}}\|\mathbf{u}'\|_{\mathrm{V}}.\end{aligned} \tag{3.7}$$

Using Temam [31, Lemma 3.4] we have

$$\begin{aligned}&\int_0^{\mathrm{T}}\big|\langle\mathbf{w}'\cdot\nabla\mathbf{u},\mathbf{u}'\rangle\big|\,\mathrm{d}t \\ &\quad\leqslant 2^{1/2}\|\mathbf{w}'\|^{1/2}_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}\|\mathbf{u}'\|^{1/2}_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}\int_0^{\mathrm{T}}\big(\|\mathbf{w}'\|_{\mathrm{V}}\|\mathbf{u}'\|_{\mathrm{V}}\big)^{1/2}\|\mathbf{u}\|_{\mathrm{V}}\,\mathrm{d}t \\ &\quad\leqslant 2^{1/2}\|\mathbf{w}'\|^{1/2}_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}\|\mathbf{u}'\|^{1/2}_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}\|\mathbf{w}'\|^{1/2}_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\|\mathbf{u}'\|^{1/2}_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})} \\ &\quad= \big(a_1\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\|\mathbf{u}'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}\big)^{1/2}\big(\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\big)^{1/2} \\ &\quad\leqslant a_1\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\|\mathbf{u}'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})} + \|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})} \\ &\quad\leqslant \frac{1}{2}\big(\epsilon_1\|\mathbf{u}'\|^2_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})} + \epsilon_2\|\mathbf{u}'\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}\big) \\ &\qquad + \frac{1}{2}\big(\epsilon_1^{-1}a_1^2 + \epsilon_2^{-1}\big)\|\mathbf{u}\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}, \quad \forall\epsilon_i > 0,\end{aligned} \tag{3.8}$$

where $a_1 = 2^{1/2}\|\mathbf{w}'\|^{1/2}_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}\|\mathbf{w}'\|^{1/2}_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}$. Integrating both sides of (3.7) from 0 to t, using (3.8) and (3.2),

$$\begin{aligned}&\big\|\mathbf{u}'(t)\big\|_0^2 + 2\int_0^t\big\|\nabla\mathbf{u}'(s)\big\|_0^2\,\mathrm{d}s \\ &\quad\leqslant \epsilon_1\|\mathbf{u}'\|^2_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})} + (\delta+\epsilon_2)\|\mathbf{u}'\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})} \\ &\qquad + \big(\epsilon_1^{-1}a_1^2 + \epsilon_2^{-1}\big)\|\mathbf{u}\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})} + \delta^{-1}\mu\|\mathbf{f}'\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})} \\ &\quad\leqslant \epsilon_1\|\mathbf{u}'\|^2_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})} + (\delta+\epsilon_2)\|\mathbf{u}'\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})} \\ &\qquad + \big(\epsilon_1^{-1}a_1^2 + \epsilon_2^{-1}\big)\|\mathbf{f}\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})} + \delta^{-1}\|\mathbf{f}'\|^2_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})}, \quad \forall\epsilon_i > 0,\ \forall\delta > 0.\end{aligned} \tag{3.9}$$

Since (3.9) is true for any $t \in [0, \mathrm{T}]$, one can show (3.3) by using the Poincaré inequality and taking ϵ_i, δ suitably in (3.9).

(iii) From (3.3), we have

$$\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{H})} \leqslant C_1\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}. \tag{3.10}$$

Using (3.6), (3.10), the intermediate space $\mathrm{V}^{-\sigma} = [\mathrm{H}, \mathrm{V}^{-1}]_\sigma$ for $0 < \sigma < 1$ and [23, Theorem 5.1],

$$\|\mathbf{u}'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-\sigma})} \leqslant C_1\|\mathbf{f}\|_{\mathrm{H}^{1-\sigma}(0,\mathrm{T};\mathrm{V}^{-1})}. \tag{3.11}$$

Using Theorem 2.1 and $\mathbf{w} \in \mathrm{L}^\infty(Q)$, for $1 < s < s_1$ the solution of (1.11) satisfies

$$\begin{aligned}
&\|\mathbf{u}(t)\|_s + \|p(t)\|_{s-1} \\
&\quad\leqslant C\left(\|\mathbf{u}'(t)\|_{s-2} + \|\mathbf{w}(t)\|_{\infty,\Omega}\|\mathbf{u}(t)\|_{s-1} + \|\mathbf{f}(t)\|_{s-2}\right) \\
&\quad\leqslant C\left(\|\mathbf{u}'(t)\|_{s-2} + \|\mathbf{u}(t)\|_{s-1} + \|\mathbf{f}(t)\|_{s-2}\right), \quad \forall \text{a.e. } t \in [0, \mathrm{T}],
\end{aligned} \tag{3.12}$$

where $C = C(\|\mathbf{w}\|_{\infty,Q})$. Squaring both sides of (3.12), integrating on the interval $[0, \mathrm{T}]$, using the fact that $\|\mathbf{u}(t)\|_{s-1} \leqslant \|\mathbf{u}(t)\|_1$ for $s < 1 + \lambda_1$, and (3.11), we obtain (3.4) with the constant C_2. □

Using Theorem 2.2 and Lemma 3.1, we can derive the following result:

Theorem 3.1. *Let $s_N < s \leqslant 2$ be given. Suppose that $\mathbf{w} \in \mathrm{L}^\infty(Q)$ and $\mathbf{w}' \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{V}) \cap \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H})$. If we assume that $\mathbf{f} \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{V}^{s-2}) \cap \mathrm{H}^1(0, \mathrm{T}; \mathrm{V}^{-1})$ and if we write the solution $[\mathbf{u}, p]$ in the form*

$$\left[\mathbf{u}(t), p(t)\right] = \sum_{j=1}^{N} (\mathcal{E} \star c_j)(t)[\Phi_j, \phi_j] + \left[\mathbf{u}_R(t), p_R(t)\right], \tag{3.13}$$

then the function $c_j \in \mathrm{H}^{(\tau-s_j)/2}(0, \mathrm{T})$ for $\tau > s_j$ and is given by

$$c_j(t) = \frac{1}{2\pi i}\int_\gamma \left\langle \Lambda_j(\lambda); (\lambda I - \mathcal{T})^{-1}\eta_j(t)\right\rangle \mathrm{d}\lambda, \tag{3.14}$$

$\Lambda_j(\lambda)$ is defined in (2.12), *η_j is the vector function given in* (3.35) *and* (3.38) *below, where γ is a vertical axis satisfying $\mathcal{R}\mathrm{e}\,\lambda < 0$, $\lambda \in \gamma$, and the remainder $[\mathbf{u}_R, p_R]$ satisfies*

$$\begin{aligned}
&\|\mathbf{u}_R\|_{\mathrm{L}^2(0,\mathrm{T};\mathbf{H}^s)} + \|p_R\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{s-1})} + \|\mathbf{u}_R'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})} + \sum_{j=1}^{N}\|c_j\|_{\mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})} \\
&\quad\leqslant C_1\left(\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})} + \|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\right)
\end{aligned} \tag{3.15}$$

where C_1 is given in Lemma 3.1. *Furthermore, if $\mathbf{f} \in \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{V}^{s-2})$, then*

$$\|\mathbf{u}_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{H}^s)} + \|\mathbf{u}'_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \|p_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}^{s-1})} + \sum_{j=1}^{N} \|c_j\|_{\mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})}$$
$$\leqslant C_2\big(\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\big), \tag{3.16}$$

where C_2 is given in Lemma 3.1.

Proof. The proof consists of three steps.

Step 1. Extending all functions by zero outside [0, T], taking the Laplace transform to (1.11) and using $\mathbf{u}(0) = 0$, we have

$$\begin{aligned}
(-\Delta + \zeta)\hat{\mathbf{u}} + \nabla \hat{p} + \mathcal{L}\{\mathbf{w}\cdot\nabla\mathbf{u}\} = \hat{\mathbf{f}} \quad &\text{in } \Omega,\\
\operatorname{div}\hat{\mathbf{u}} = 0 \quad &\text{in } \Omega,\\
\hat{\mathbf{u}} = 0 \quad &\text{on } \Gamma.
\end{aligned} \tag{3.17}$$

From Theorem 2.2, we see that if $\hat{\mathbf{f}} \in \mathrm{V}^{s-2}$ for $s_N < s \leqslant 2$, then the solution $[\hat{\mathbf{u}}, \hat{p}]$ of (3.17) can be written by: for number ζ with $\mathcal{R}\mathrm{e}\,\zeta > 0$,

$$[\hat{\mathbf{u}}, \hat{p}] = \sum_{j=1}^{N} c_j^\zeta e^{-r\sqrt{\zeta}} [\Phi_j, \phi_j] + [\hat{\mathbf{u}}_R, \hat{p}_R] \tag{3.18}$$

and the coefficient c_j^ζ is given by $c_j^\zeta = \Lambda_j^\zeta[\mathbf{h}, 0]$ with $\mathbf{h} = \mathcal{L}\{\mathbf{f} - \mathbf{w}\cdot\nabla\mathbf{u}\}$, satisfies the inequality (2.31) and the remainder $[\hat{\mathbf{u}}_R, \hat{p}_R]$ satisfies the inequality (2.34). Moreover the coefficient function $c_j^\zeta e^{-r\sqrt{\zeta}}$ is the Laplace transform of a convolution in the time variable t:

$$c_j^\zeta e^{-r\sqrt{\zeta}} = \mathcal{L}\big\{(\mathcal{E} \star c_j)(t)\big\}(z), \tag{3.19}$$

where $\mathcal{E}(\mathbf{x}, t) = \mathcal{L}^{-1}\{e^{-r\sqrt{\zeta}}\}$ and $c_j(t) = \mathcal{L}^{-1}\{c_j^\zeta\}$. Explicitly, $\mathcal{E}(\mathbf{x}, t) = re^{-r^2/4t}\big/\sqrt{4\pi t^3}$ and

$$\begin{aligned}
c_j(t) &= \mathcal{L}^{-1}\big\{\Lambda_j(\zeta)\big[\mathbf{h}(\zeta), 0\big]\big\}(t) \quad \big(\Lambda_j(\zeta) = \Lambda_j^\zeta\big)\\
&= \mathcal{L}^{-1}\left\{\frac{1}{2\pi i}\int_\gamma \frac{\Lambda_j(\lambda)}{\lambda - \zeta}\,\mathrm{d}\lambda\big[\mathbf{h}(\zeta), 0\big]\right\}(t)\\
&= \frac{1}{2\pi i}\int_\gamma \Lambda_j(\lambda)\mathcal{L}^{-1}\left\{\frac{[\mathbf{h}(\zeta), 0]}{\lambda - \zeta}\right\}(t)\,\mathrm{d}\lambda \quad \big(\mathcal{T} = [\partial_t, 0]\big)\\
&= \frac{1}{2\pi i}\int_\gamma \big\langle \Lambda_j(\lambda); (\lambda I - \mathcal{T})^{-1}\big[\mathcal{L}^{-1}\{\mathbf{h}\}(t), 0\big]\big\rangle\,\mathrm{d}\lambda,
\end{aligned} \tag{3.20}$$

where γ is a vertical axis satisfying $\mathcal{R}\mathrm{e}\,\lambda < 0$, $\lambda \in \gamma$. Extending c_j by zero outside (0, T) and using (2.31), we have, for $s_j < s \leqslant 2$,

$$
\begin{aligned}
\|c_j\|_{\mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})} &= \big\|\big(1+|\zeta|\big)^{(s-s_j)/2}\Lambda_j(\zeta)\big(\mathbf{h}(\zeta),0\big)\big\|_{\mathrm{L}^2(-\infty,\infty)} \\
&\leqslant C\left(\int_{-\infty}^{\infty}\big\|\mathbf{h}(\zeta)\big\|_{s-2}^2\,\mathrm{d}\zeta\right)^{1/2} \\
&\leqslant C\left(\int_0^{\mathrm{T}}\|\mathbf{f}-\mathbf{w}\cdot\nabla\mathbf{u}\|_{s-2}^2\,\mathrm{d}t\right)^{1/2} \\
&\leqslant C\big(\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}+\|\mathbf{w}\|_{\infty,Q}^2\|\mathbf{u}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-1})}\big) \\
&\leqslant C\big(\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}+\|\mathbf{w}\|_{\infty,Q}\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{-1})}\big) \\
&\leqslant C\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})},
\end{aligned}
$$

where $C = C(\mathcal{T}, \|\mathbf{w}\|_{\infty,Q})$.

Taking the inverse Laplace transform to (3.18)–(3.19) and using (3.20), we obtain the decomposition (3.13). In Step 3 we will show that the function $c_j(t)$ given in (3.20) can be expressed in terms of given data.

Step 2. We show the inequalities (3.15) and (3.16). Inserting the remainder $[\mathbf{u}_R, p_R]$ of (3.13) into equations in (1.11), we have

$$
\begin{aligned}
\partial_t\mathbf{u}_R-\Delta\mathbf{u}_R+(\mathbf{w}\cdot\nabla)\mathbf{u}_R+\nabla p_R &= \mathbf{F} \quad \text{in } Q, \\
\operatorname{div}\mathbf{u}_R &= g_s \quad \text{in } Q, \\
\mathbf{u}_R = 0 \quad \text{on } \Sigma, \qquad \mathbf{u}_R(\cdot,0) &= 0 \quad \text{in } \Omega,
\end{aligned} \tag{3.21}
$$

where $\mathbf{F} = \mathbf{f} + \mathbf{f}_s$ with

$$
\begin{aligned}
\mathbf{f}_s &= -\sum_{i=1}^{N}\big[\partial_t(\mathcal{E}\Phi_i)-\Delta(\mathcal{E}\Phi_i)+\mathbf{w}\cdot\nabla(\mathcal{E}\Phi_i)+\nabla(\mathcal{E}\phi_i)\big]\star c_i, \\
g_s &= -\sum_{i=1}^{N}\operatorname{div}(\mathcal{E}\Phi_i)\star c_i.
\end{aligned}
$$

Using Lemma 3.1(ii), the remainder $\mathbf{u}_R(t) := \mathbf{u}(t) - \sum_{j=1}^N(\mathcal{E}\star c_j)(t)\Phi_j$ satisfies

$$
\begin{aligned}
\|\mathbf{u}_R'\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}+\|\mathbf{u}_R'\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})} &\leqslant C_1\left(\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}+\sum_{j=1}^{N}\|c_j\|_{\mathrm{L}^2[0,\mathrm{T}]}\right) \\
&\leqslant C_1\big(\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}+\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}\big),
\end{aligned} \tag{3.22}
$$

where C_1 is defined in Lemma 3.1(ii) and $s_N < s \leqslant 2$. Using Theorem 2.2 the solution $[\mathbf{u}_R, p_R]$ of (3.21) satisfies the inequality: for $s_N < s \leqslant 2$,

$$
\begin{aligned}
&\big\|\mathbf{u}_R(t)\big\|_s + \big\|p_R(t)\big\|_{s-1} \\
&\quad \leqslant C\big(\big\|\mathbf{u}'_R(t)\big\|_{s-2} + \big\|\mathbf{f}(t) - \mathbf{w}(t)\cdot\nabla\mathbf{u}_R(t)\big\|_{s-2} + \|\mathbf{f}_s\|_{s-2} + \big\|g_s(t)\big\|_{s-1}\big) \\
&\quad \leqslant C\left(\big\|\mathbf{u}'_R(t)\big\|_{s-2} + \big\|\mathbf{w}(t)\cdot\nabla\mathbf{u}_R(t)\big\|_{s-2} + \big\|\mathbf{f}(t)\big\|_{s-2} + \sum_{j=1}^{N}\|c_j\|_{\mathrm{L}^2(0,\mathrm{T})}\right) \\
&\quad \leqslant C\big(\big\|\mathbf{u}'_R(t)\big\|_{s-2} + \big\|\mathbf{u}_R(t)\big\|_{s-1} + \big\|\mathbf{f}(t)\big\|_{s-2} + \|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}\big),
\end{aligned}
\tag{3.23}
$$

where $C = C(\|\mathbf{w}\|_{\infty,Q})$. Squaring both sides of (3.23), integrating on the interval $[0,\mathrm{T}]$ and using Lemma 3.1(iii), the inequality (3.15) follows. Finally, if $s \leqslant 2$, then $\|\mathbf{u}'_R(t)\|_{s-2} \leqslant \|\mathbf{u}'_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H})}$, and since $\mathbf{u}_R(t) = \int_0^t \mathbf{u}'_R(s)\,\mathrm{d}s$, $\|\mathbf{u}_R(t)\|_{s-1} \leqslant \sqrt{\mathrm{T}}\|\mathbf{u}'_R\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V})}$. Using (3.22)–(3.23), we have

$$
\begin{aligned}
&\big\|\mathbf{u}_R(t)\big\|_s + \big\|p_R(t)\big\|_{s-1} \\
&\quad \leqslant C_2\big(\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})} + \|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})}\big)
\end{aligned}
\tag{3.24}
$$

where $C_2 = C(C_1, \|\mathbf{w}\|_{\infty,Q})$. Thus (3.16) follows by (3.24).

Step 3. We show that the coefficient function $c_j(t)$ given in (3.20) can be expressed explicitly in terms of data. We define

$$
\mathcal{A}_i[\mathbf{f}, g] := \frac{1}{2\pi i}\int_\gamma \big\langle \Lambda_i(\lambda) : (\lambda I - \mathcal{T})^{-1}[\mathbf{f}, g]\big\rangle\,\mathrm{d}\lambda. \tag{3.25}
$$

For $i = 1, \ldots, N$, let

$$
\begin{aligned}
\mathbf{u}_{i,s} = (\mathcal{E}\star c_i)\Phi_i, \qquad & p_{i,s} = (\mathcal{E}\star c_i)\phi_i, \\
\big[\Phi_i^*, \phi_i^*\big] = (\mathcal{T} + \mathrm{L} + B_{\mathbf{w}})&[\mathcal{E}\Phi_i, \mathcal{E}\phi_i].
\end{aligned}
\tag{3.26}
$$

(a) Writing $[\mathbf{u}_1, p_1] = [\mathbf{u}, p] - [\mathbf{u}_{1,s}, p_{1,s}]$ and using (1.13), we have

$$
\begin{aligned}
(\mathcal{T} + \mathrm{L} + B_{\mathbf{w}})[\mathbf{u}_1, p_1] = [\mathbf{f}, 0] - \big[\Phi_1^*, \phi_1^*\big]\star c_1 \quad &\text{in } Q, \\
\mathbf{u}_1 = 0 \quad &\text{on } \Sigma, \\
\mathbf{u}_1(\cdot, 0) = 0 \quad &\text{in } \Omega.
\end{aligned}
\tag{3.27}
$$

Since the first leading corner singularity has been already subtracted in the remainder $[\mathbf{u}_1, p_1]$, we must have

$$
\mathcal{A}_1\big([\mathbf{f}, 0] - \big[\Phi_1^*, \phi_1^*\big]\star c_1 - B_{\mathbf{w}}[\mathbf{u}_1, p_1]\big) = 0. \tag{3.28}
$$

To express the remainder $[\mathbf{u}_1, p_1]$ in terms of data, we set

$$
\begin{aligned}
\mathcal{S} &= (\mathcal{T} + \mathrm{L})^{-1}, \\
\mathcal{S}_{\mathbf{w}} &= (I + \mathcal{S}B_{\mathbf{w}})^{-1}, \\
J_{\mathbf{w}} &= I - B_{\mathbf{w}}\mathcal{S}_{\mathbf{w}}\mathcal{S}.
\end{aligned}
\tag{3.29}
$$

Since $\mathcal{S}$ is the solution operator of the evolution Stokes problem, it is bounded, so the operators $\mathcal{S}_{\mathbf{w}}$ and $J_{\mathbf{w}}$ are bounded under the assumption of $\mathbf{w}$. Using (3.29), the solution of (3.27) is

$$[\mathbf{u}_1, p_1] = \mathcal{S}_{\mathbf{w}}\mathcal{S}\big([\mathbf{f}, 0] - \big[\Phi_1^*, \phi_1^*\big] \star c_1\big). \tag{3.30}$$

Using (3.30) and $J_{\mathbf{w}}$,

$$[\mathbf{f}, 0] - \big[\Phi_1^*, \phi_1^*\big] \star c_1 - B_{\mathbf{w}}[\mathbf{u}_1, p_1] = J_{\mathbf{w}}\big([\mathbf{f}, 0] - \big[\Phi_1^*, \phi_1^*\big] \star c_1\big), \tag{3.31}$$

and Eq. (3.28) becomes

$$c_1 \star \mathcal{A}_1\big(J_{\mathbf{w}}\big[\Phi_1^*, \phi_1^*\big]\big) = \mathcal{A}_1\big(J_{\mathbf{w}}[\mathbf{f}, 0]\big). \tag{3.32}$$

Extending by zero outside $[0, \mathrm{T}]$ and taking the Laplace transform to (3.32),

$$\begin{aligned}\hat{c}_1(z) &= \Lambda_1^z(\alpha_1)/\Lambda_1^z(\beta_1),\\ \alpha_1 &:= \mathcal{L}\big\{J_{\mathbf{w}}[\mathbf{f}, 0]\big\},\\ \beta_1 &:= \mathcal{L}\big\{J_{\mathbf{w}}\big[\Phi_1^*, \phi_1^*\big]\big\}.\end{aligned} \tag{3.33}$$

Next we claim that $\Lambda_1^z(\beta_1) \neq 0$. It is enough to show that $J_{\mathbf{w}}$ is one-to-one and

$$\big[\Phi_1^*, \phi_1^*\big] \neq 0.$$

Indeed, let $[\mathbf{v}, q] = J_{\mathbf{w}}[\mathbf{f}, 0]$ for $\mathbf{f} \neq 0$. If $[\mathbf{v}, q] = 0$, then $0 = (I - B_{\mathbf{w}}\mathcal{S}_{\mathbf{w}}\mathcal{S})[\mathbf{f}, 0]$ and $B_{\mathbf{w}}\mathcal{S}_{\mathbf{w}}\mathcal{S}[\mathbf{f}, 0] = [\mathbf{f}, 0]$. Let $[\mathbf{u}, p] = \mathcal{S}_{\mathbf{w}}\mathcal{S}[\mathbf{f}, 0]$. Then $B_{\mathbf{w}}[\mathbf{u}, p] = [\mathbf{f}, 0]$ and $(I + \mathcal{S}B_{\mathbf{w}})[\mathbf{u}, p] = \mathcal{S}[\mathbf{f}, 0]$. Hence $[\mathbf{u}, p] = \mathcal{S}([\mathbf{f}, 0] - B_{\mathbf{w}}[\mathbf{u}, p]) = \mathcal{S}[0, 0]$. So $[\mathbf{u}, p] = [0, 0]$ and $\mathbf{f} = 0$, which contradicts with $\mathbf{f} \neq 0$. So $J_{\mathbf{w}}$ is 1–1. If $[\Phi_1^*, \phi_1^*] = [0, 0]$, then

$$[0, 0] = \big[\Phi_1^*, \phi_1^*\big] = (\mathcal{T} + \mathrm{L} + \mathcal{B}_{\mathbf{w}})[\mathcal{E}\Phi_1, \mathcal{E}\phi_1]$$

and $(I + \mathcal{S}B_{\mathbf{w}})[\mathcal{E}\Phi_1, \mathcal{E}\phi_1] = [0, 0]$. Since $I + \mathcal{S}B_{\mathbf{w}}$ is 1–1, $[\mathcal{E}\Phi_1, \mathcal{E}\phi_1] = [0, 0]$, which is impossible. We conclude that $J_{\mathbf{w}}[\Phi_1^*, \phi_1^*] \neq [0, 0]$ and $\Lambda_1^z(\beta_1) \neq 0$. Thus the number $\hat{c}_1(z)$ in (3.33) is well defined, and using (2.31),

$$\big|\hat{c}_1(z)\big| \leqslant C\big|\Lambda_1^z\big[\alpha_1(z)\big]\big| \leqslant C\big\|\alpha_1(z)\big\|_{s-2}\big(1 + |z|\big)^{(s_1 - s)/2}, \tag{3.34}$$

so $\|c_1\|_{\mathrm{H}^{(s-s_1)/2}(0,\mathrm{T})} \leqslant C\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathbf{H}^{s-2})}$ for $s_1 < s < s_2$, where $C = C(\|J_{\mathbf{w}}\|)$. In particular, the stress intensity function c_1 is given by

$$\begin{aligned}c_1(t) &= \mathcal{L}^{-1}\big\{\Lambda_1^z(\alpha_1)/\Lambda_1^z(\beta_1)\big\}(t)\\ &= \mathcal{L}^{-1}\big\{\Lambda_1^z\big(\alpha_1/\Lambda_1^z(\beta_1)\big)\big\}(t)\\ &= \frac{1}{2\pi i}\int_{\gamma}\big\langle \Lambda_1^{\lambda} : (\lambda I - \mathcal{T})^{-1}\eta_1(t)\big\rangle\, \mathrm{d}\lambda,\end{aligned} \tag{3.35}$$

where $\eta_1 := \mathcal{L}^{-1}\{\alpha_1/\Lambda_1^z(\beta_1)\}$.

(b) If $N = 1$, our assertion is done in Step 1. Let $N = 2$. Writing $[\mathbf{u}_R, p_R] = [\mathbf{u}, p] - [\mathbf{u}_{2,s}, p_{2,s}] - [\mathbf{u}_{1,s}, p_{1,s}]$, we have

$$(\mathcal{T} + \mathrm{L} + B_{\mathbf{w}})[\mathbf{u}_R, p_R] = [\mathbf{f}, 0] - \left[\Phi_1^*, \phi_1^*\right] \star c_1 - \left[\Phi_2^*, \phi_2^*\right] \star c_2 \quad \text{in } Q,$$

with boundary conditions $\mathbf{u}_R = 0$ on Σ and $\mathbf{u}_R(\cdot, 0) = 0$ in Ω. Using exactly the same procedures as used in Step 1, one has

$$\begin{aligned} \hat{c}_2(z) &= \Lambda_2^z(\alpha_2)/\Lambda_2^z(\beta_2), \\ \alpha_2 &:= \alpha_1 - \hat{c}_1(z)\beta_1, \\ \beta_2 &:= \mathcal{L}\left\{J_{\mathbf{w}}\left[\Phi_2^*, \phi_2^*\right]\right\} \neq 0. \end{aligned} \tag{3.36}$$

Like $\Lambda_1^z(\beta_1) \neq 0$, we have $\Lambda_2^z(\beta_2) \neq 0$, and since $\beta_1 \in \mathbf{H}^{s-2}$ for $s > s_2$, using (2.31),

$$\begin{aligned} \left|\hat{c}_2(z)\right| &\leqslant C\left|\Lambda_2^z(\alpha_2)\right| \\ &\leqslant C\left(\left\|\Lambda_2^z(\alpha_1)\right\|_{s-2} + \left|\hat{c}_1(z)\right|\left\|\Lambda_2^z(\beta_1)\right\|_{s-2}\right) \\ &\leqslant C\left(\left\|\alpha_1(z)\right\|_{s-2} + \left|\hat{c}_1(z)\right|\left\|\beta_1(z)\right\|_{s-2}\right)\left(1 + |z|\right)^{(s_2-s)/2} \\ &\leqslant C\left(\left\|\alpha_1(z)\right\|_{s-2} + \left|\hat{c}_1(z)\right|\right)\left(1 + |z|\right)^{(s_2-s)/2} \end{aligned} \tag{3.37}$$

and

$$\|c_2\|_{\mathrm{H}^{(s-s_2)/2}(0,\mathrm{T})} \leqslant C\|\mathbf{f}\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{V}^{s-2})} \quad (s > s_2),$$

where $C = C(\|J_{\mathbf{w}}\|)$. Similarly,

$$c_2(t) = \frac{1}{2\pi i}\int_\gamma \left\langle \Lambda_2^\lambda : (\lambda I - \mathcal{T})^{-1}\eta_2(t)\right\rangle \mathrm{d}\lambda, \tag{3.38}$$

where $\eta_2 := \mathcal{L}^{-1}\{\alpha_2/\Lambda_2^z(\beta_2)\}$. $\square$

4. The nonlinear problem

In this section we study the problem (1.1) on the region $Q = \Omega \times (0, \mathrm{T})$ where Ω is a bounded polygon having only one concave vertex, say the origin. Let $N = 1$ or 2, depending on the opening angle ω of the origin (see (1.5)–(1.6)). We show that the solution $[\mathbf{u}, p]$ of problem (1.1) on the region Q can have a corner singularity expansion in the following sense: if we write the solution $[\mathbf{u}, p]$ of (1.1) in the form

$$[\mathbf{u}, p] = [\mathbf{u}_R, p_R] + \sum_{j=1}^N (\mathcal{E} \star c_j)[\Phi_j, \phi_j],$$

the remainder $[\mathbf{u}_R, p_R]$ satisfies an increased regularity in a certain Banach space, and the stress intensity functions c_j are well defined, can be expressed in terms of known data.

First we introduce vector notations of the singular functions, their coefficients, and their multiplications as follows:

$$\mathcal{P} = [\Phi_1, \dots, \Phi_N]^T, \qquad \phi = [\phi_1, \dots, \phi_N]^T, \qquad \mathbf{c} = [c_1, \dots, c_N]^T,$$
$$(\mathcal{E} \star \mathbf{c})\Phi = \sum_{j=1}^{N} (\mathcal{E} \star c_j)\Phi_j, \tag{4.1}$$

where Φ_j, ϕ_j are given in (1.4) and $\mathcal{E}$ is defined in (1.7). We also denote some notations for spaces and their norms as follows: for $s_N < s \leqslant 2$,

$$\begin{aligned}
&\mathbf{X} = \mathrm{L}^\infty\big(0, \mathrm{T}; \mathbf{H}^s \cap \mathrm{V}\big) \cap \mathrm{H}^1(0, \mathrm{T}; \mathrm{V}) \cap \mathrm{H}^{1,\infty}(0, \mathrm{T}; \mathrm{H}),\\
&Y = \mathrm{L}^\infty\big(0, \mathrm{T}; \mathrm{H}^{s-1}\big),\\
&\tilde{\mathbf{X}} = \mathbf{X} \times Z, \quad Z = \prod_{j=1}^{N} \mathrm{H}^{(s-s_j)/2}(0, \mathrm{T}) \quad (s > s_j),\\
&\big\|[\mathbf{v}, \mathbf{d}]\big\|_{\tilde{\mathbf{X}}} := \|\mathbf{v}\|_{\mathbf{X}} + \|\mathbf{d}\|_Z,\\
&\|\mathbf{d}\|_Z := \sum_{j=1}^{N} \|d_j\|_{\mathrm{H}^{(s-s_j)/2}(0,\mathrm{T})}.
\end{aligned} \tag{4.2}$$

Define

$$\begin{aligned}
&[\mathbf{u}_s, p_s] = (\mathcal{E} \star \mathbf{c})[\Phi, \phi],\\
&[\mathbf{f}_s, g_s] = [\mathcal{T} + \mathrm{L} + B_{\mathbf{w}}][\mathbf{u}_s, p_s],\\
&\mathbf{c} = [c_1, \dots, c_N],\\
&c_j = \frac{1}{2\pi i} \int_\gamma \big\langle \Lambda_j^\lambda : (\lambda I - \mathcal{T})^{-1} \eta_j(t) \big\rangle \, \mathrm{d}\lambda,
\end{aligned} \tag{4.3}$$

where η_j is defined in (3.35) and (3.38). We define a ball B_a as follows: for number $0 < a \leqslant 1$,

$$B_a = \big\{[\mathbf{w}_R, \mathbf{d}] \in \tilde{\mathbf{X}}\colon \ \big\|[\mathbf{w}_R, \mathbf{d}]\big\|_{\tilde{\mathbf{X}}} \leqslant a\big\}.$$

Let $\mathbf{w}$ be a fixed vector function defined by

$$\mathbf{w} = \mathbf{w}_R + (\mathcal{E} \star \mathbf{d})\Phi, \quad \mathbf{w}_R \in \mathbf{X},\ \mathbf{d} \in Z. \tag{4.4}$$

Here we define a mapping T on the ball B_a as follows: for fixed $\mathbf{f}$,

$$B_a \ni [\mathbf{w}_R, \mathbf{d}] \to \mathrm{T}[\mathbf{w}_R, \mathbf{d}] = [\mathbf{u}_R, \mathbf{c}]$$

where $\mathbf{c}$ is a vector function with the components c_j defined in (3.11) and $[\mathbf{u}_R, p_R]$ is the solution of the problem

$$[\mathcal{T}+\mathrm{L}+B_{\mathbf{w}}][\mathbf{u}_R,p_R]=[\mathbf{f},0]-[\mathbf{f}_s,g_s]\quad\text{in } Q,$$
$$\mathbf{u}=0\quad\text{on }\Sigma,$$
$$\mathbf{u}(\cdot,0)=0\quad\text{in }\Omega, \tag{4.5}$$

where $\mathbf{f}_s$ and g_s are defined in (4.3).

Using Theorem 3.1 we can derive the following a priori estimate for the solution of (4.5):

Lemma 4.1. *Let $s_N<s\leqslant 2$ be given. Let $[\mathbf{w}_R,\mathbf{d}]\in B_a$. Let $\mathbf{w}=\mathbf{w}_R+\mathbf{w}_s$ with $\mathbf{w}_s=(\mathcal{E}\star\mathbf{d})\Phi$. If $\mathbf{f}\in \mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})\cap\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})$, then the pair $[\mathbf{u}_R,p_R,\mathbf{c}]$ satisfies*

$$\begin{aligned}&\big\|[\mathbf{u}_R,\mathbf{c}]\big\|_{\tilde{\mathbf{X}}}+\|p_R\|_Y\\&\quad\leqslant C\big(\big\|[\mathbf{w}_R,\mathbf{d}]\big\|_{\tilde{\mathbf{X}}}\big\|[\mathbf{u}_R,\mathbf{c}]\big\|_{\tilde{\mathbf{X}}}+\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})}+\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\big),\end{aligned}\tag{4.6}$$

where $C=C(\|J_{\mathbf{w}}\|)$.

Proof. Using the solution operator $\mathcal{S}=(\mathcal{T}+\mathrm{L})^{-1}$,

$$\begin{aligned}[\mathbf{u}_R,p_R]&=\mathcal{S}\big([\mathbf{f},0]-B_{\mathbf{w}}[\mathbf{u}_R,p_R]-[\mathbf{f}_s,g_s]\big)\\&=\mathcal{S}\big([\mathbf{f},0]-B_{\mathbf{w}}[\mathbf{u}_R,p_R]-\mathcal{B}_{\mathbf{w}}[\mathbf{u}_s,p_s]-[\mathbf{u}_s',0]-\mathrm{L}[\mathbf{u}_s,p_s]\big).\end{aligned}\tag{4.7}$$

So

$$[\mathbf{u}_R,p_R]=\mathcal{S}\big(\big[\mathbf{f}-\mathbf{u}_s'-\mathbf{w}\cdot\nabla(\mathbf{u}_R+\mathbf{u}_s),0\big]-\mathrm{L}[\mathbf{u}_s,p_s]\big).$$

Like (3.22)–(3.24), the pair $[\mathbf{u}_R,p_R]$ can be estimated by

$$\begin{aligned}&\big\|\mathbf{u}_R(t)\big\|_s+\big\|p_R(t)\big\|_{s-1}\\&\quad\leqslant C\big(\big\|\mathbf{w}\cdot\nabla(\mathbf{u}_R+\mathbf{u}_s)\big\|_{s-2}+\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}+\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})}\big)\end{aligned}\tag{4.8}$$

for all $t\in[0,\mathrm{T}]$ where $C=C(\|\mathcal{S}\|)$. Since $\mathbf{w}_s\in\mathbf{H}^\sigma(\Omega)$ for any $\sigma<s_1$, $\mathbf{w}(\cdot,t)$ is continuous on the closure of Ω and for $s_N<s\leqslant 2$,

$$\begin{aligned}\big\|\mathbf{w}\cdot\nabla(\mathbf{u}_R+\mathbf{u}_s)\big\|_{s-2}&\leqslant\|\mathbf{w}\|_{\infty,\Omega}\big\|\nabla(\mathbf{u}_R+\mathbf{u}_s)\big\|_{s-2}\\&\leqslant\big(\|\mathbf{w}_R\|_s+\|\mathbf{w}_s\|_{\infty,\Omega}\big)\big(\|\mathbf{u}_R\|_{s-1}+\|\mathbf{u}_s\|_{s-1}\big)\\&\leqslant C\big\|[\mathbf{w}_R,\mathbf{d}]\big\|_{\tilde{\mathbf{X}}}\big(\|\mathbf{u}_R\|_{s-1}+\|\mathbf{u}_s\|_{s-1}\big)\end{aligned}\tag{4.9}$$

where the last inequality follows by $|(\mathcal{E}(r,\cdot)\star\mathbf{d})(t)|\leqslant C\|\mathbf{d}\|_{\mathrm{L}^2(0,\mathrm{T})}$. In order to compute $\|\mathbf{u}_s\|_{s-1}$ we will use the following equivalence of the seminorm $|u|_\sigma$ for $0<\sigma<1$:

$$|u|_\sigma^2:=\int\limits_\Omega\int\limits_\Omega\frac{|u(\mathbf{x})-u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2\sigma}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\sim|u|_r^2+|u|_\theta^2,$$

where

$$|u|_r^2 := \int_{\omega_1}^{\omega_2}\int_0^1\int_0^1 \frac{|u(r,\theta)-u(r_1,\theta)|^2}{|r-r_1|^{1+2\sigma}} r\,\mathrm{d}r\,\mathrm{d}r_1\mathrm{d}\theta,$$

$$|u|_\theta^2 := \int_0^1\int_{\omega_1}^{\omega_2}\int_{\omega_1}^{\omega_2} \frac{|u(r,\theta)-u(r,\theta_1)|^2}{|\theta-\theta_1|^{1+2\sigma}} r^{1-2\sigma}\,\mathrm{d}\theta\,\mathrm{d}\theta_1\,\mathrm{d}r.$$

Take $\sigma = s-1$ where $s_N < s < 2$. Set $k = r^{\lambda_1}\mathcal{E}$. Then

$$(\mathcal{E}\star c_1)\Phi_1 = (k\star c_1)\chi\mathcal{T}_1$$

and

$$\begin{aligned} k(r,t)-k(r_1,t) &= \left(r^{\lambda_1}-r_1^{\lambda_1}\right)\mathcal{E}(r,t) + r_1^{\lambda_1}\left(\mathcal{E}(r,t)-\mathcal{E}(r_1,t)\right) \\ &= \left(r^{\lambda_1}-r_1^{\lambda_1}\right)\mathcal{E}(r,t) + r_1^{\lambda_1}(r-r_1)\mathcal{E}_r(r_*,t) \end{aligned}$$

for r_* between r and r_1. Using the Young's theorem [1, Theorem 4.30, p. 90],

$$\begin{aligned} \left|\mathbf{u}_s(t)\right|_r^2 &\leqslant C\int_0^1\int_0^1 |r-r_1|^{-(1+2\sigma)}\left|\kappa(r,t)-\kappa(r_1,t)\right|^2 r\,\mathrm{d}r\,\mathrm{d}r_1 \|c_1\|^2_{\mathrm{L}^2(0,\mathrm{T})} \\ &\leqslant C\int_0^1\int_0^1 \frac{|r^{\lambda_1}-r_1^{\lambda_1}|^2}{|r-r_1|^{1+2\sigma}}\mathcal{E}(r,t)^2 r\,\mathrm{d}r\,\mathrm{d}r_1\|d_1\|^2_{\mathrm{L}^2(0,\mathrm{T})} \\ &\quad + C\int_0^1\int_0^1 \frac{|r-r_1|^2}{|r-r_1|^{1+2\sigma}}\mathcal{E}_r(r_*,t)^2 r_1^{2\lambda_1+1}\,\mathrm{d}r\,\mathrm{d}r_1\,\|c_1\|^2_{\mathrm{L}^2(0,\mathrm{T})}. \end{aligned}$$

Since $\mathcal{E}(r,t)<\infty$ and $r^{\lambda_1}\in \mathrm{H}^\sigma(\Omega)$, we have

$$\left|\mathbf{u}_s(t)\right|_r \leqslant C\|c_1\|_{\mathrm{L}^2(0,\mathrm{T})}$$

for a constant C. Since $\mathcal{T}_1(\theta)$ is a smooth function of θ, we have $|\mathbf{u}_s(t)|_\theta \leqslant C\|c_1\|_{\mathrm{L}^2(0,\mathrm{T})}$. Hence

$$\left\|(\mathcal{E}\star c_1)\Phi_1\right\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{H}^{s-1})} \leqslant C\|c_1\|_{\mathrm{L}^2(0,\mathrm{T})},$$

where C is a constant. Similarly for $(\mathcal{E}\star c_2)\Phi_2$. So

$$\left\|(\mathcal{E}\star \mathbf{c})\Phi\right\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{H}^{s-2})} \leqslant C\|\mathbf{c}\|_{\mathrm{L}^2(0,\mathrm{T})}. \tag{4.10}$$

From (4.9) and (4.10),

$$\left\|\mathbf{w}\cdot\nabla(\mathbf{u}_R+\mathbf{u}_s)\right\|_{s-2} \leqslant C\left\|[\mathbf{w}_R,\mathbf{d}]\right\|_{\tilde{\mathbf{X}}}\left\|[\mathbf{u}_R,\mathbf{c}]\right\|_{\tilde{\mathbf{X}}} \tag{4.11}$$

for a constant C. This inequality easily follows for $s=2$. Combining (4.8) and (4.11),

$$\begin{aligned}&\|\mathbf{u}_R\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathbf{H}^s)}+\|p_R\|_Y\\&\quad\leqslant C\big(\big\|[\mathbf{w}_R,\mathbf{d}]\big\|_{\tilde{\mathbf{X}}}\big\|[\mathbf{u}_R,\mathbf{c}]\big\|_{\tilde{\mathbf{X}}}+\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}+\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})}\big)\end{aligned}\tag{4.12}$$

where $C=C(\|\mathcal{S}\|)$. From (3.16) we have $\|\mathbf{c}\|_Z\leqslant C(\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})}+\|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})})$ where $C=C(\|J_{\mathbf{w}}\|)$. Thus, using (4.12) and (3.22), the required inequality (4.6) follows. □

Lemma 4.2. *For fixed* $\mathbf{f}$*,* T *is a contraction on* B_a *in the topology of* $\tilde{\mathbf{X}}$ *if* a *is small enough.*

Proof. Let $\mathbf{w}=\mathbf{w}_R+(\mathcal{E}\star\mathbf{d})\Phi$ and $\mathbf{w}^*=\mathbf{w}_R^*+(\mathcal{E}\star\mathbf{d}^*)\Phi$. For fixed $\mathbf{f}$ we consider $\mathrm{T}[\mathbf{w}_R,\mathbf{d}]=[\mathbf{u}_R,\mathbf{c}]$ and $\mathrm{T}[\mathbf{w}_R^*,\mathbf{d}^*]=[\mathbf{u}_R^*,\mathbf{c}^*]$ where $[\mathbf{u}_R,p_R]$ and $[\mathbf{u}_R^*,p_R^*]$ are the solutions of (4.3), respectively. Set $\mathbf{u}_1:=\mathbf{u}_R-\mathbf{u}_R^*$, $p_1:=p_R-p_R^*$, $\mathbf{w}_1:=\mathbf{w}_R-\mathbf{w}_R^*$, $\mathbf{c}_1:=\mathbf{c}-\mathbf{c}^*$ and $\mathbf{d}_1:=\mathbf{d}-\mathbf{d}^*$. Then

$$\begin{aligned}\partial_t\mathbf{u}_1-\Delta\mathbf{u}_1+\mathbf{w}\cdot\nabla\mathbf{u}_1+\nabla p_1&=\mathbf{k}+\mathbf{k}_s\quad\text{in } Q,\\ \operatorname{div}\mathbf{u}_1&=g_s\quad\text{in } Q,\\ \mathbf{u}_1&=0\quad\text{on } \Sigma,\\ \mathbf{u}_1(\cdot,0)&=0\quad\text{in } \Omega,\end{aligned}\tag{4.13}$$

where

$$\begin{aligned}&\mathbf{k}=(\mathbf{w}_1+\mathbf{d}_1\star\mathcal{E}\Phi)\cdot\nabla\big[\mathbf{w}_R^*+(\mathcal{E}\star\mathbf{d}^*)\Phi\big]-\mu^{-1}\big(\mathbf{w}_R+(\mathcal{E}\star\mathbf{d})\Phi\big)\cdot\nabla(\mathbf{d}_1\star\mathcal{E}\Phi),\\&\mathbf{k}_s=\big[\partial_t(\mathcal{E}\Phi)-\Delta(\mathcal{E}\Phi)+\nabla(\mathcal{E}\phi)\big]\star\mathbf{c}_1,\\&g_s=\mathbf{c}_1\star\operatorname{div}(\mathcal{E}\Phi).\end{aligned}\tag{4.14}$$

If $[\mathbf{w}_R,\mathbf{d}]$ and $[\mathbf{w}_R^*,\mathbf{d}^*]$ are in B_a, the function $\mathbf{k}$ can be easily estimated by

$$\|\mathbf{k}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})}+\|\mathbf{k}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\leqslant Ca\big\|[\mathbf{w}_1,\mathbf{d}_1]\big\|_{\tilde{\mathbf{X}}}.\tag{4.15}$$

Like Lemma 4.1 and using (4.15), the solution pair $[\mathbf{u}_1,\mathbf{c}_1]$ of (4.13) satisfies

$$\begin{aligned}\big\|[\mathbf{u}_1,\mathbf{c}_1]\big\|_{\tilde{\mathbf{X}}}+\|p_1\|_Y&\leqslant C\big\|[\mathbf{w},\mathbf{d}]\big\|_{\tilde{\mathbf{X}}}\big\|[\mathbf{u}_1,\mathbf{c}_1]\big\|_{\tilde{\mathbf{X}}}+Ca\big\|[\mathbf{w}_1,\mathbf{d}_1]\big\|_{\tilde{\mathbf{X}}}\\&\leqslant Ca\big\|[\mathbf{u}_1,\mathbf{c}_1]\big\|_{\tilde{\mathbf{X}}}+Ca\big\|[\mathbf{w}_1,\mathbf{d}_1]\big\|_{\tilde{\mathbf{X}}},\end{aligned}\tag{4.16}$$

where $C=C(\|J_{\mathbf{w}}\|)$. If a is sufficiently small such that $Ca<1/3$ in (4.16), then

$$\big\|[\mathbf{u}_1,\mathbf{c}_1]\big\|_{\tilde{\mathbf{X}}}\leqslant\big\|[\mathbf{u}_1,\mathbf{c}_1]\big\|_{\tilde{\mathbf{X}}}+\|p_1\|_Y\leqslant\frac{1}{2}\big\|[\mathbf{w}_1,\mathbf{d}_1]\big\|_{\tilde{\mathbf{X}}}.\tag{4.17}$$

Hence T is a contraction on B_a in the topology of $\tilde{\mathbf{X}}$. □

We now give the proof of Theorem 1.2. Set $\tilde{\mathbf{u}}_j=[\mathbf{u}_{R,j},\mathbf{c}_j]$. Let $\tilde{\mathbf{u}}_0\in B_a$ be given. Define $\tilde{\mathbf{u}}_j=\mathrm{T}\tilde{\mathbf{u}}_{j-1}$ for $j=1,2,\ldots.$ Since T is a contraction on B_a, the sequence $\{\tilde{\mathbf{u}}_j\}$ is a Cauchy sequence in $\tilde{\mathbf{X}}$. Since $\tilde{\mathbf{X}}=\mathbf{X}\times Z$ is complete, there is an element $\tilde{\mathbf{u}}=[\mathbf{u}_R,\mathbf{c}]\in\tilde{\mathbf{X}}$ such that

$\|\tilde{\mathbf{u}}_j - \tilde{\mathbf{u}}\|_{\tilde{\mathbf{X}}} \to 0$ as $j \to \infty$. Since T is continuous on B_a, the limit $\tilde{\mathbf{u}}$ satisfies $\tilde{\mathbf{u}} = \mathrm{T}\tilde{\mathbf{u}}$. From (4.6) there is a bounded sequence $\{p_{R,j}\} \subset Y$ and an element $p_R \in Y$ such that $\|p_{R,j} - p_R\|_Y \to 0$ as $j \to \infty$.

Let $\tilde{\mathbf{u}}_j \in B_a$ and $p_{R,j} \in Y$. We define

$$\begin{aligned}
&\mathbf{u}_j = \mathbf{u}_{R,j} + (\mathcal{E} \star \mathbf{c}_j)\Phi,\\
&p_j = p_{R,j} + (\mathcal{E} \star \mathbf{c}_j)\phi,\\
&\mathbf{c}_j = [c_{1,j}, \ldots, c_{N,j}], \quad 1 \leqslant N \leqslant 2,\\
&c_{k,j} = \frac{1}{2\pi i}\int_\gamma \big\langle \Lambda_k^\lambda : (\lambda I - \mathcal{T})^{-1}\eta_{k,j}(t)\big\rangle \,\mathrm{d}\lambda \quad (1 \leqslant k \leqslant N),
\end{aligned}$$

where $\eta_{k,j} = \mathcal{L}^{-1}\{\alpha_{k,j}/\Lambda_k^z(\beta_{k,j})\}$ with

$$\begin{aligned}
&\alpha_{1,j} = \mathcal{L}\big\{J_{\mathbf{u}_j}[\mathbf{f}, 0]\big\}, \qquad \beta_{1,j} = \mathcal{L}\big\{J_{\mathbf{u}_j}\big[\Phi_{1,j}^*, \phi_1^*\big]\big\},\\
&\alpha_{2,j} = \alpha_{1,j} - \hat{c}_{1,j}(z)\beta_{1,j}, \qquad \beta_{2,j} = \mathcal{L}\big\{J_{\mathbf{u}_j}\big[\Phi_{2,j}^*, \phi_2^*\big]\big\},\\
&\big[\Phi_{k,j}^*, \phi_k^*\big] = (\mathcal{T} + \mathrm{L} + B_{\mathbf{u}_j})[\mathcal{E}\Phi_k, \mathcal{E}\phi_k].
\end{aligned}$$

If $\tilde{\mathbf{u}}_j \to \tilde{\mathbf{u}} := [\mathbf{u}_R, \mathbf{c}]$, we have $\mathbf{u}_j \to \mathbf{u} = \mathbf{u}_R + (\mathcal{E} \star \mathbf{c})\Phi$, so $B_{\mathbf{u}_j} \to B_{\mathbf{u}}$ and the operators defined in (3.29) satisfy

$$\mathcal{S}_{\mathbf{u}_j} \to \mathcal{S}_{\mathbf{u}}, \qquad J_{\mathbf{u}_j} \to J_{\mathbf{u}}.$$

Using these we see that $c_{k,j} \to c_k$ because $\alpha_{k,j} \to \alpha_k$, $\beta_{k,j} \to \beta_k$, and $\eta_{k,j} \to \eta_k$ as $j \to \infty$. Also, if $p_{R,j} \to p_R$, then $p_j \to p = p_R + (\mathcal{E} \star \mathbf{c})\phi$. Finally it remains to show that the limit triple $[\mathbf{u}_R, p_R, \mathbf{c}]$ solves the equations of (4.5) with $B_{\mathbf{w}} = B_{\mathbf{u}}$. It is sufficient to show that

$$\langle B_{\mathbf{u}_j}\mathbf{u}_j - B_{\mathbf{u}}\mathbf{u}, \mathbf{v}\rangle \to 0, \quad \forall \mathbf{v} \in \mathbf{H}_0^1.$$

For $s_N < s \leqslant 2$, we have

$$\begin{aligned}
\|B_{\mathbf{u}_j}\mathbf{u}_j - B_{\mathbf{u}}\mathbf{u}\|_{-1} &\leqslant \|B_{\mathbf{u}_j}\mathbf{u}_j - B_{\mathbf{u}}\mathbf{u}\|_{s-2}\\
&\leqslant \big\|(\mathbf{u}_j - \mathbf{u}) \cdot \nabla\mathbf{u}_j\big\|_{s-2} + \big\|\mathbf{u} \cdot \nabla(\mathbf{u}_j - \mathbf{u})\big\|_{s-2}\\
&\leqslant C\big(\|\tilde{\mathbf{u}}_j - \tilde{\mathbf{u}}\|_{\tilde{\mathbf{X}}}\|\tilde{\mathbf{u}}_j\|_{\tilde{\mathbf{X}}} + \|\tilde{\mathbf{u}}\|_{\tilde{\mathbf{X}}}\|\tilde{\mathbf{u}}_j - \tilde{\mathbf{u}}\|_{\tilde{\mathbf{X}}}\big)\\
&\to 0 \quad \text{as } j \to \infty \text{ and for all } t \in [0, \mathrm{T}].
\end{aligned}$$

Clearly we have $\|(\mathcal{T} + \mathrm{L})[\mathbf{u}_j - \mathbf{u}, p_j - p]\|_{-1} \to 0$ as $j \to \infty$. Hence the limit triple $[\mathbf{u}_R, p_R, \mathbf{c}]$ solves Eqs. (1.1) and from Lemma 4.1, satisfies

$$\big\|[\mathbf{u}_R, \mathbf{c}]\big\|_{\tilde{\mathbf{X}}} + \|p_R\|_Y \leqslant C\big(\|\mathbf{f}\|_{\mathrm{L}^\infty(0,\mathrm{T};\mathrm{V}^{s-2})} + \|\mathbf{f}\|_{\mathrm{H}^1(0,\mathrm{T};\mathrm{V}^{-1})}\big). \tag{4.18}$$

Thus Theorem 1.2 is shown.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] J.D. Anderson Jr., Fundamentals of Aerodynamics, second ed., McGraw–Hill, New York, 1991.
[3] L.C. Evans, Partial Differential Equations, Amer. Math. Soc., 1998.
[4] M. Dauge, Stationary Stokes and Navier–Stokes systems on two- or three-dimensional domains with corners. Part I: Linearized equations, SIAM J. Math. Anal. 20 (1989) 74–97.
[5] P. Deuring, The Stokes System in an Infinite Cone, Mathematical Research, vol. 78, Akademie-Verlag, Berlin, 1994.
[6] P. Deuring, L^p-theory for the Stokes system in 3D domains with conical boundary points, Indiana Univ. Math. J. 47 (1998) 11–47.
[7] P. Deuring, The Stokes resolvent in 3D domains with conincal boundary points: Nonregularity in L^p-spaces, Adv. Differential Equations 6 (2001) 175–228.
[8] M.G. Garroni, V.A. Solonnikov, M.A. Vivaldi, Existence and regularity results for oblique derivative problems for heat equations in an angle, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998) 47–79.
[9] V. Girault, P.-A. Raviart, Finite Element Methods for Navier–Stokes Equations: Theory Algorithms, Springer-Verlag, 1986.
[10] P. Grisvard, Elliptic Problems in Non-Smooth Domains, Pitman, Boston, 1985.
[11] P. Grisvard, Edge behavior of the solution of an elliptic problem, Math. Nachr. 132 (1987) 281–299.
[12] P. Grisvard, Singularities in Boundary Value Problems, RMA, vol. 22, Springer-Verlag, New York, 1992.
[13] P. Grisvard, Singular behavior of elliptic problems in non-Hilbertian Sobolev spaces, J. Math. Pures Appl. 74 (1995) 3–33.
[14] R.B. Kellogg, J.E. Osborn, A regularity result for the Stokes problem in a convex domain, J. Funct. Anal. 21 (1976) 397–431.
[15] V.A. Kondratév, Boundary Problems for Elliptic Equations with Conical or Angular Points, Trans. Moscow Math. Soc., vol. 16, 1967, translated by Amer. Math. Soc., 1998.
[16] V.A. Kozlov, V.G. Maz'ya, Singularities of solutions of the first boundary value problem for the heat equation in domains with conical points, I, Izv. Vyssh. Uchebn. Zaved. Mat. 84 (2) (1987) 38–46.
[17] V.A. Kozlov, V.G. Maz'ya, Singularities of solutions of the first boundary value problem for the heat equation in domains with conical points, II, Izv. Vyssh. Uchebn. Zaved. Mat. 84 (3) (1987) 38–46.
[18] V.A. Kozlov, Coefficients in the asymtotics of the solutions of initial boundary value parabolic problems in domains with a conic point, Siberian Math. J. 29 (1988) 222–233.
[19] V.A. Kozlov, V.G. Maz'ya, J. Rossmann, Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations, Amer. Math. Soc., 2001.
[20] J.R. Kweon, R.B. Kellogg, Compressible Stokes problem on non-convex polygon, J. Differential Equations 176 (2001) 290–314.
[21] J.R. Kweon, R.B. Kellogg, Regularity of solutions to the Navier–Stokes equations for compressible barotropic flows on a polygon, Arch. Ration. Mech. Anal. 163 (2002) 35–64.
[22] J.R. Kweon, An evolution compressible Stokes system in a polygon, J. Differential Equations 199 (2004) 352–375.
[23] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, I, Springer-Verlag, Berlin, 1972.
[24] A.A. Manuel, J.O. Paulo, T.P. Fernando, Benchmark solutions for the flow of Oldroyd-B and PTT fluids in planar contractions, J. Non-Newtonian Fluid Mech. 110 (2003) 45–75.
[25] A.I. Nazarov, L^p estimates for a solution to the Dirichlet problem and to the Neumann problem for the heat equation in a wedge with edge of arbitrary codimension, J. Math. Sci. (N.Y.) 106 (2001) 2989–3014.
[26] A.I. Nazarov, Dirichlet problem for quasilinear parabolic equations in domains with smooth closed edges, in: Proceedings of the St. Petersburg Mathematical Society, vol. IX, in: Amer. Math. Soc. Transl. Ser. 2, vol. 209, Amer. Math. Soc., Providence, RI, 2003, pp. 115–141.
[27] S.A. Nazarov, A. Novotny, K. Pileckas, On steady compressible Navier–Stokes equations in plane domains with corners, Math. Ann. 304 (1996) 1, 121–150.
[28] S.A. Nazarov, B.A. Plamenevsky, Elliptic Problems in Domains with Piecewise Smooth Boundaries, Walter de Gruyter, Berlin, 1994.
[29] P.N. Shankar, M.D. Deshpande, Fluid mechanics in the driven cavity, Annu. Rev. Fluid Mech. 32 (2000) 93–136.
[30] V.A. Solonnikov, L^p-estimates for solutions of the heat equation in a dihedral angle, Rend. Mat. Appl. (7) 21 (2001) 1–15.
[31] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, North-Holland, Amsterdam, 1985.
[32] M.F. Webster, H.R. Tamaddon-Jahromi, M. Aboubacar, Transient viscoelastic flows in planar contractions, J. Non-Newtonian Fluid Mech. 118 (2004) 83–101.